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INTERMEDIATE
TRIGONOMETRY

(Covering Pre-University, Higher Secondary & Entrance Courses)

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PRE-UNIVERSITY
INTERMEDIATE
TRIGONOMETRY

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PREFACE

WE have revised our book Intermediate Trigonometry in accordance with the syllabus of Pre-University course of the Calcutta and Pre-University and Entrance course of other Universities, so that the present book is intended to be a text book for the Pre-University and Entrance students. The book also covers completely the syllabus of Higher Secondary Course of the Board of Secondary Education, West Bengal. We have tried to make the exposition clear and concise without going into unnecessary details. A good number of examples has been worked out by way of illustration and examples set have been carefully selected.

Important formulae and results have been given at the beginning of the book for reference. Some typical question papers are given at the end to give the students an idea of the standard of the examination.

It is hoped that the book will meet the requirements of those for whom it is intended and we shall deem our labours amply rewarded if the students find the book useful to them.

Any criticisms, corrections and suggestions towards improvement will be thankfully received.

CALCUTTA
June 1933

B. C. D.
B. N. M.

GREEK LETTERS USED IN THE BOOK

α	(Alpha)	β	(Bēta)	γ	(Gamma)
δ	(Delta)	θ	(Thetā)	π	(Pai)
ϕ	(Phai)	ψ	(Psi)	Δ	(Delta)

Note. The notation C. U. used at the end of any example means that the example was set at Intermediate Examination of the Calcutta University.

SYLLABUS

for

Pre University and Entrance Courses

TRIGONOMETRY

Measurement of angles. Sexagesimal and circular measure.

Definitions of trigonometrical ratios, their mutual relations.

Deduction of the values of the trigonometrical ratios of 0°, 30°, 45°, 60°, 90°.

Trigonometrical ratios of associated angles.

Addition and subtraction formulae.

Transformation of products and sums of trigonometrical ratios.

Multiple and sub-multiple angles (simple cases).

General values, Solution of trigonometrical equations.

Inverse circular functions.

Trigonometrical identities.

Relation between sides and angles of a triangle, area, in-radius and circum-radius of a triangle.

Solution of triangles with use of log tables.

Graphs of simple trigonometrical functions.

Simple problems of heights and distances.

Syllabus for the Higher Secondary Course

TRIGONOMETRY

Class IX

Measurement of angles in degrees, minutes, seconds and in radians. Definition of trigonometrical ratios of an acute angle. Trigonometrical ratios of the standard angles—0°, 30°, 45°, 60°, 90°, (undefined values such as tan 90°, cot 0° to be excluded). Simple identities connecting the ratios of an angle immediately derivable from a right-angled Triangle. Trigonometrical ratios of complementary angles.

Easy problems on heights and distances reducible to the solution of right-angled triangles involving the standard angles above.

Class X

Trigonometrical ratios of any angle; Trigonometrical ratios of angles associated with a given angle, Addition and subtraction formulae, Transformation of products and sums; Multiple and sub-multiple angles.

Class XI

Graphs of simple trigonometric functions.

Trigonometric equations and general values · Inverse Circular Functions.

Relation between sides and angles of a triangle; In-radius, circum-radius and area of a triangle; Practical solution of a triangle with the help of logarithms; Simple problems of heights and distances.

CONTENTS

IMPORTANT FORMULÆ AND RESULTS

A radian = $57° \ 17' \ 44{\cdot}8''$ nearly.

1 degree = $\cdot 01745$ radians nearly.

2 right angles = $180°$ = π radians.

$\pi = \frac{22}{7} = 3{\cdot}1416$ approximately.

Radian measure of an angle at the centre of a circle

$$= \frac{\text{subtending arc}}{\text{radius}}.$$

II. $\sin^2\theta + \cos^2\theta = 1$,

$\sec^2\theta = 1 + \tan^2\theta$,

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\frac{\cos\theta}{\sin\theta} = \cot\theta.$$

III.

$\sin 0° = 0$,	$\cos 0° = 1$,	$\tan 0° = 0$.
$\sin 30° = \frac{1}{2}$;	$\cos 30° = \frac{\sqrt{3}}{2}$;	$\tan 30° = \frac{1}{\sqrt{3}}$.
$\sin 45° = \frac{1}{\sqrt{2}}$;	$\cos 45° = \frac{1}{\sqrt{2}}$;	$\tan 45° = 1$.
$\sin 60° = \frac{\sqrt{3}}{2}$;	$\cos 60° = \frac{1}{2}$;	$\tan 60° = \sqrt{3}$.
$\sin 90° = 1$,	$\cos 90° = 0$,	$\tan 90° = \infty$.
$\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$;	$\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$,	$\tan 15° = 2 - \sqrt{3}$.
$\sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}}$;	$\cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$;	$\tan 75° = 2 + \sqrt{3}$.
$\sin 18° = \frac{1}{4}(\sqrt{5}-1)$,	$\cos 36° = \frac{1}{4}(\sqrt{5}+1)$.	
$\sin 120° = \frac{\sqrt{3}}{2}$,	$\cos 120° = -\frac{1}{2}$.	
$\sin 180° = 0$,	$\cos 180° = -1$	$\tan 180° = 0$.
$\sin 270° = -1$,	$\cos 270° = 0$,	$\tan 270° = \infty$.
$\sin 360° = 0$,	$\cos 360° = 1$;	$\tan 360° = 0$.

IV. $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$.

$\sin(90^\circ - \theta) = \cos\theta$; $\quad \sin(90^\circ + \theta) = \cos\theta$.

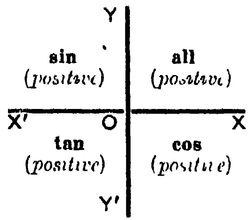
$\cos(90^\circ - \theta) = \sin\theta$; $\quad \cos(90^\circ + \theta) = -\sin\theta$.

$\tan(90^\circ - \theta) = \cot\theta$; $\quad \tan(90^\circ + \theta) = -\cot\theta$.

$\sin(180^\circ - \theta) = \sin\theta$; $\quad \sin(180^\circ + \theta) = -\sin\theta$.

$\cos(180^\circ - \theta) = -\cos\theta$, $\quad \cos(180^\circ + \theta) = -\cos\theta$.

$\tan(180^\circ - \theta) = -\tan\theta$; $\quad \tan(180^\circ + \theta) = \tan\theta$.



V. $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}.$$

VI. $2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \sin B = \sin(A+B) - \sin(A-B)$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$.

VII. $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$.

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$$

$$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}.$$

VIII. $\sin 2A = 2 \sin A \cos A$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}; \quad \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\left.\begin{aligned} 1 - \cos 2A &= 2 \sin^2 A \\ 1 + \cos 2A &= 2 \cos^2 A \end{aligned}\right\}$$

$$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}$$

IX. $\sin 3A = 3 \sin A - 4 \sin^3 A$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

X. $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$

$$\cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} = 2 \cos^2 \frac{\theta}{2} - 1 = 1 - 2 \sin^2 \frac{\theta}{2}$$

$$\tan \theta = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$$

$$\sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}; \quad \cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$$

$$1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$$

$$1 + \cos \theta = 2 \cos^2 \frac{\theta}{2}$$

$$\frac{1 - \cos \theta}{1 + \cos \theta} = \tan^2 \frac{\theta}{2}.$$

XI. If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$.

If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$.

If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$.

If $\sin\theta = 0$, or, $\tan\theta = 0$, $\theta = n\pi$.

If $\cos\theta = 0$, or, $\cot\theta = 0$, $\theta = (2n+1)\dfrac{\pi}{2}$.

If $\sin\theta = 1$, $\theta = (4m+1)\dfrac{\pi}{2}$; if $\sin\theta = -1$, $\theta = (4m-1)\dfrac{\pi}{2}$.

If $\cos\theta = 1$, $\theta = 2m\pi$; if $\cos\theta = -1$, $\theta = (2m+1)\pi$.

XII. $\sin^{-1}x + \cos^{-1}x = \frac{1}{2}\pi$

$\tan^{-1}x + \cot^{-1}x = \frac{1}{2}\pi$

$\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{1}{2}\pi$

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$

$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$

$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\dfrac{x+y+z-xyz}{1-yz-zx-xy}$

$\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\}$

$\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\{xy \mp \sqrt{1-x^2}.\sqrt{1-y^2}\}$.

XIII. $\log_a mn = \log_a m + \log_a n$

$\log_a \dfrac{m}{n} = \log_a m - \log_a n$; $\log_a m^n = n\log_a m$;

$\log_a m = \log_b m \times \log_a b$; $\log_a 1 = 0$; $\log_a a = 1$.

XIV. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$$\left.\begin{aligned} \cos A &= \frac{b^2+c^2-a^2}{2bc}; \\ \cos B &= \frac{c^2+a^2-b^2}{2ca}; \\ \cos C &= \frac{a^2+b^2-c^2}{2ab}. \end{aligned}\right\}$$

$$a = b \cos C + c \cos B.$$
$$b = c \cos A + a \cos C.$$
$$c = a \cos B + b \cos A.$$

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$$

$$\sin B = \frac{2}{ca} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{ca}$$

$$\sin C = \frac{2}{ab} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{ab}.$$

$$\begin{aligned}\triangle &= \tfrac{1}{2}bc \sin A = \tfrac{1}{2}ca \sin B = \tfrac{1}{2}ab \sin C \\ &= \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } 2s = a + b + c \\ &= \frac{abc}{4R}.\end{aligned}$$

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} = \frac{abc}{4\Delta}.$$

$$\begin{aligned}r &= \frac{\Delta}{s} = 4R \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \\ &= (s-a) \tan \tfrac{1}{2}A = (s-b) \tan \tfrac{1}{2}B = (s-c) \tan \tfrac{1}{2}C.\end{aligned}$$

$$\begin{aligned}r_1 &= \frac{\Delta}{s-a} = 4R \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C \\ &= s \tan \tfrac{1}{2}A.\end{aligned}$$

$$\begin{aligned}r_2 &= \frac{\Delta}{s-b} = 4R \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B \cos \tfrac{1}{2}C \\ &= s \tan \tfrac{1}{2}B.\end{aligned}$$

$$\begin{aligned}r_3 &= \frac{\Delta}{s-c} = 4R \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \sin \tfrac{1}{2}C \\ &= s \tan \tfrac{1}{2}C.\end{aligned}$$

IMPORTANT RESULTS

1. If $A+B+C=\pi$, then

(i) $\sin A+\sin B+\sin C=4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$.

(ii) $\cos A+\cos B+\cos C=1+4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$.

(iii) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

(iv) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$.

(v) $\cos 2A+\cos 2B+\cos 2C$

$$=-4\cos A\cos B\cos C-1.$$

(vi) $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1$.

(vii) $\cot B\cot C+\cot C\cot A+\cot A\cot B=1$.

(viii) $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}$

$$=1+4\sin\frac{B+C}{4}\sin\frac{C+A}{4}\sin\frac{A+B}{4}.$$

(ix) $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}$

$$=4\cos\frac{B+C}{4}\cos\frac{C+A}{4}\cos\frac{A+B}{4}.$$

(x) $\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1$.

(xi) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.

2. $\underset{\theta\to 0}{Lt}\frac{\sin\theta}{\theta}=1$, $\underset{\theta\to 0}{Lt}\cos\theta=1$, $\underset{\theta\to 0}{Lt}\frac{\tan\theta}{\theta}=1$.

3. Area of a circle of radius $r=\pi r^2$.

Perimeter of a circle of radius $r=2\pi r$.

PRE-UNIVERSITY INTERMEDIATE TRIGONOMETRY

CHAPTER I

MEASUREMENT OF ANGLES

1. TRIGONOMETRY, as indicated by its very name, originally meant a subject which dealt with the methods of measurement of triangles. At present its scope has widened, and now it means a subject which deals with the measurements relating to any angle, not necessarily an angle of a triangle.

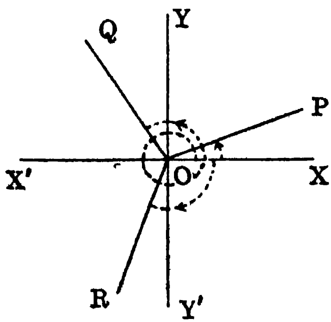
2. Angles in Trigonometry.

In Geometry, angles are supposed to be formed by the intersection of two straight lines and are always restricted to lie between 0° and 360°, being acute, obtuse or reflex. Moreover, they are always positive, negative angles having no meaning. In Trigonometry however, the idea of an angle is much more general.

An angle in Trigonometry is supposed to be formed by the revolution of a straight line which starts from an initial position coinciding with one arm, and traces out the angle by its revolution about one extremity until it reaches the final position coinciding with the other arm.

For instance, the angle XOP is formed by the revolution of a line which starts from the initial position OX, and revolving in the anti-clockwise direction, traces out the angle XOP which is acute. The same line again, starting from OX and revolving in the anti-clockwise direction may make a complete revolution and further move up to the position OQ. The angle formed in this case is more than

five right angles. Now revolutions may be clockwise or anti-clockwise. *It is conventional to consider angles formed by the anti-clockwise revolution of the revolving line to be positive.* Angles formed by clockwise revolutions of the



revolving line will then be considered *negative angles.* For example, the angle XOR measured in the clockwise direction from the initial position OX is a negative angle.

Thus, *angles in Trigonometry may be of any magnitude and may be positive as well as negative.*

OX being the initial position of the revolving line, produce XO to X', and let YOY' be the perpendicular line. The whole plane is thus divided into four quadrants, the first being XOY, the second YOX', the third $X'OY'$ and the fourth $Y'OX$. If we contemplate an angle say $+920^\circ$ to be traced out by the revolving line, the line must have completed two complete revolutions, thereby describing $2 \times 360^\circ = 720^\circ$, and have further traced out an angle 200°, so that the final position of the revolving line is in the third quadrant. Similarly, if we consider an angle -1354°, the final position of the revolving line is in the first quadrant, for $-1354^\circ = -360^\circ \times 3 - 274^\circ$.

It should be noted that if two angles differ by complete multiples of 360°, the starting line being the same, the final

position of the revolving line will be coincident for the two angles. For example, the angles $255°$ and $-105°$ will have the final positions of the revolving line same, if both start from the same initial position.

3. Units of measurement of angles.

We should now define the different systems of units used for the measurement of angles. In defining a unit however, a standard angle, which has no reference to any particular system of unit, should form the basis, and such a standard angle is a right angle. A right angle is defined in books on Geometry to be an angle which any straight line standing on another makes with it, when the two adjacent angles formed are equal to one another. A right angle is always the same everywhere, and it thus forms a suitable basis to start with, in defining the different systems of measurement of angles.

There are three systems of units used in Trigonometry for measurement of angles, *viz.*,

(i) Sexagesimal unit.

(ii) Centesimal unit.

(iii) Circular unit.

Sexagesimal* System. In this system, a right angle is divided into 90 equal parts, each being called a *degree*. A degree is again divided into 60 sexagesimal *minutes*, and each minute is further subdivided into 60 sexagesimal *seconds*, so that

1 rt. angle = $90°$ (degrees)

$1°$ = $60'$ (sexagesimal minutes)

$1'$ = $60''$ (sexagesimal seconds)

* So called, since the subdivisions are mostly by sixtieth parts. It is also called the *Common* or the *English System*.

Centesimal† System. In this system, the subdivisions of a right angle are as follows :

1 rt. angle $=100^g$ (grades)

1^g $=100'$ (centesimal minutes)

$1'$ $=100''$ (centesimal seconds).

Note. It may be noted that $1'$ (centesimal minute) is not the same as $1'$ (sexagesimal minute), the former being $\frac{1}{100\times 100}$ of a right angle and the latter being $\frac{1}{90\times 60}$ of a right angle, so that the first is $\frac{27}{50}$th part of the second. Similarly, $1''$ is less than $1''$, being only $\frac{81}{250}$th part of it.

The connection between the two systems of units may be effected through a right angle, remembering that 1 right angle $=90^\circ=100^g$, so that $9^\circ=10^g$. Any angle in the first system may be reduced to degrees, and then multiplied by $\frac{10}{9}$ will be reduced to grades. Similarly, an angle in the second system may be changed to the first.

We shall presently deal with the third system, namely the circular system.

4. Theorem. *In all circles, the circumference bears a constant ratio to its diameter.*

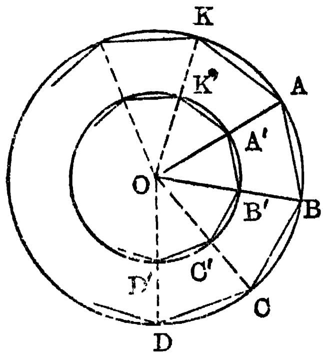
Take any two circles of any radii, and place them with a common centre O. In one, let $ABCD\ldots$ be an inscribed regular polygon of n sides. Let $A', B', C', \ldots$ be the points of intersection of the radii $OA, OB, OC, \ldots$ with the other circle. It is easily seen that $A'B'C'\ldots$ is also a regular polygon of n sides, inscribed in the second circle. Now $OA=OB$, as also $OA'=OB'$, so that in the triangles OAB, $OA'B'$,

† So called because the subdivisions are by hundredths. It is also called the *French System*.

$OA : OA' = OB : OB'$, and angle O is common. The two triangles are therefore similar. Hence, $AB : A'B' = OA : OA'$.

Thus,

$$\frac{\text{perimeter of polygon } ABCD\ldots}{\text{perimeter of polygon } A'B'C'D'\ldots} = \frac{n.AB}{n.A'B'} = \frac{OA}{OA'}.$$



This being true, whatever the number of sides n may be, making n infinitely large, the perimeters of the polygons can be made practically coincident with the circumferences of the corresponding circles, and thus we deduce that

$$\frac{\text{circumference of the circle } ABCD\ldots}{\text{circumference of the circle } A'B'C'D'\ldots} = \frac{OA}{OA'},$$

$$i.e. \quad = \frac{\text{radius of circle } ABC\ldots}{\text{radius of circle } A'B'C'\ldots},$$

Thus circumference of any circle : its radius is the same for all circles. As diameter is twice the radius, we deduce that the circumference of any circle bears a constant ratio to its diameter.

This constant ratio is denoted by the Greek letter π. Its actual value has been determined by methods which are outside the scope of the present book, by some mathematicians

to more than 500 places of decimals. An approximate value commonly used is $\frac{22}{7}$. A more accurate value is $\frac{355}{113}$

Expressed in decimal, the value is nearly 3·14159...

Hence, if r be the radius of a circle, d its diameter,

the circumference $= \pi d = 2\pi r$.

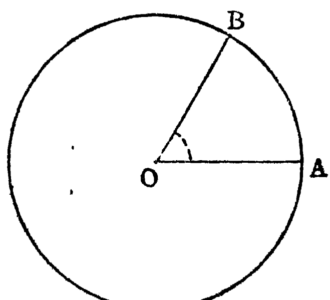
where $\pi = 3{\cdot}14159... = \frac{22}{7}$ roughly.

5. Circular Unit or Radian Measure.

In any circle, if we take an arc whose length is equal to the radius of the circle, the angle which this arc subtends at the centre is called a *radian*, and is written as 1^c.

We shall now show that with reference to whichever circle it may be defined, a radian is a constant angle, and hence it may be used as a suitable unit for measurement of angles, which is known as the circular unit.

Theorem I. *A radian is a constant angle.*



Let AB be an arc of any circle with centre O whose length is equal to its radius OA. By definition, $\angle AOB =$ 1 radian. Since angles at the centre of a circle are proportional to the arcs which subtend them, and the whole angle

round O subtended by the complete circumference being known from Geometry to be 4 right angles, we get,

$$\frac{\angle AOB}{4 \text{ right angles}} = \frac{\text{arc } AB}{\text{whole circumference}} = \frac{\text{radius}}{\text{circumference}}$$

i.e., $\frac{1 \text{ radian}}{4 \text{ rt. } \angle^s} = \frac{r}{2\pi r} = \frac{1}{2\pi}$, r being the radius.

Hence, $1 \text{ radian} = \frac{2}{\pi} \text{ rt. angle.}$

$\therefore$ a radian is a constant angle. (π being constant)

Note. We thus see that whatever be the radius of the circle with reference to which a radian is defined, its magnitude is the same.

From above, π **radians = 180°.**

$\therefore$ **1 radian** $= \frac{180}{\pi} = \frac{180}{3 \cdot 14159} = 57 \cdot 29577$ degrees

$= 57°17'44 \cdot 8''$ nearly.

$\therefore$ **1 degree = ·0174533 radians** nearly.

In higher mathematics so far as theoretical investigations are concerned, as a matter of convenience, angles are usually measured in the circular unit, *i.e.*, in radians. In this connection we may state the following theorem :

Theorem II. *The measure of any angle in radians is expressed by the ratio of the arc of any circle subtending that angle at its centre, to the radius.*

Let XOP be any angle.

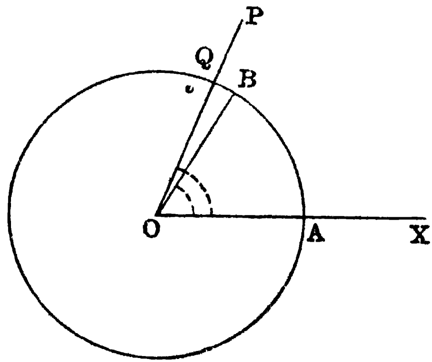
With centre O and any radius OA draw a circle, and let AQ be the arc which subtends the angle XOP at the centre O. Let AB be the arc whose length is equal to the radius AO; so that, by definition, $\angle AOB$ is one radian.

Now from Geometry, angles at the centre of a circle are proportional to the arcs which subtend them.

Hence, $\dfrac{\angle XOP}{\angle AOB} = \dfrac{\text{arc } AQ}{\text{arc } AB} = \dfrac{\text{arc } AQ}{\text{radius } OA},$

or, $\dfrac{\angle XOP}{1 \text{ radian}} = \dfrac{\text{arc } AQ}{\text{radius } OA},$

i.e., $\angle XOP = \dfrac{\text{arc } AQ}{\text{radius } OA}$ of a radian.



Thus, if θ be the radian-measure of the $\angle XOP$, s be the length of the arc AQ, and r the radius of the circle, then

$$\theta = \frac{s}{r} \quad \text{or,} \quad s = r\theta.$$

Note. In higher mathematics, *when an angle is expressed in radian measure, the unit is generally implied and not expressed,* so that, when the measure of an angle is given without the unit being mentioned, we should always understand it to be in radians. For example, 'an angle is $\frac{\pi}{2}$' means that the angle is $\frac{\pi}{2}$ radians, which converted to degrees is $90°$ *i.e.*, one right angle.

6. In working out examples, relations between the three systems of units should be carefully remembered, namely

$$\textbf{1 rt.}\angle = 90° = 100^g = \frac{\pi}{2} \textbf{ radians},$$

whence, $\pi^c = 180°$.

Ex. 1. *Express*

(i) $63^\circ\ 22'\ 40{\cdot}8''$ *in centesimal measure*

and (ii) $203^g\ 58^{\backprime}\ 73^{\backprime\backprime}$ *in radians.*

Here (i) $63^\circ\ 22'\ 40{\cdot}8'' = 63\frac{189}{500}$ deg. $= \frac{31689}{500} \times \frac{1}{90}$ rt. $\angle$

$$= \frac{31689}{500} \times \frac{1}{90} \times 100 \text{ grades} = \frac{3521}{50} \text{ grades}$$

$$= 70^g\ 42^{\backprime}.$$

(ii) $203^g\ 58^{\backprime}\ 73^{\backprime\backprime} = 203{\cdot}5873$ grades

$$= 2{\cdot}035873 \text{ rt.} \angle = 2{\cdot}035873 \times \frac{\pi}{2} \text{ radians}$$

$$= 1{\cdot}0179365\ \pi \text{ radians.}$$

Ex. 2. *Two angles of a triangle are* $72^\circ\ 53'\ 51''$, *and* $41^g\ 22^{\backprime}\ 50^{\backprime\backprime}$ *respectively. Find the third angle in radians.*

$$41^g\ 22^{\backprime}\ 50^{\backprime\backprime} = 41{\cdot}2250 \text{ grades}$$

$$= \frac{41{\cdot}225 \times 9}{10} \text{ degrees } [\, 9^\circ = 10^g \,]$$

$$= 37{\cdot}1025 \text{ degrees}$$

$$= 37^\circ\ 6'\ 9''.$$

The sum of the two given angles is therefore

$$72^\circ\ 53'\ 51'' + 37^\circ\ 6'\ 9'' = 110^\circ.$$

The sum of the three angles of a triangle being 180°, the third angle is

$$180^\circ - 110^\circ = 70^\circ = 70^\circ \times \frac{\pi}{180} \text{ radians} \quad [\, \pi^c = 180^\circ \,]$$

$$= \frac{7\pi}{18} \text{ radians.}$$

Ex. 3. *Divide* $\frac{\pi}{4}$ *radians into two parts such that the number of sexagesimal minutes in one may be to the number of centesimal seconds in the other part as* 27 : 2500.

We have $\frac{\pi}{4}$ radians $= \frac{\pi}{4} \times \frac{2}{\pi}$ rt. $\angle = \frac{1}{2}$ rt. $\angle$.

Let x be the number of centesimal seconds in the second part, so that $\frac{27}{2500}x$ is the number of sexagesimal minutes in the first part.

Now $$x'' = \frac{x}{100 \times 100 \times 100} \text{ rt. } \angle$$

and $$\frac{27}{2500}x' = \frac{27x}{2500 \times 60 \times 90} \text{ rt. } \angle = \frac{x}{500000} \text{ rt. } \angle$$

$$\therefore \quad \frac{x}{1000000} + \frac{x}{500000} = \frac{1}{2},$$

whence $x = \frac{500000}{3}$.

Thus, second part is $\frac{500000''}{3} = \frac{500000}{3 \times 100 \times 100 \times 100} \text{ rt.} \angle$

$= \frac{1}{6}$ rt.$\angle = 15°$, and as the sum of the two parts is $\frac{1}{2}$ rt.$\angle$ *i.e.*, $45°$, the first part is $30°$.

The two parts are therefore $30°$ and $15°$.

Ex. 4. *The angles of a quadrilateral are in A.P., and the number of grades in the least angle is to the number of radians in the greatest as* $100 : \pi$. *Find the angles in degrees.*

Let the angles, expressed in degrees, be α, $\alpha + \beta$, $\alpha + 2\beta$ and $\alpha + 3\beta$ respectively. Then

$$\alpha + \alpha + \beta + \alpha + 2\beta + \alpha + 3\beta = 360, \qquad \cdots \quad (1)$$

i.e., $2\alpha + 3\beta = 180$.

Again the least angle, $\alpha° = \frac{10}{9}\alpha^g$

and the greatest angle $(\alpha + 3\beta)° = (\alpha + 3\beta)\frac{\pi}{180}^c$

and so from the given condition,

$$\frac{10}{9}\alpha/(\alpha + 3\beta)\frac{\pi}{180} = 100/\pi,$$

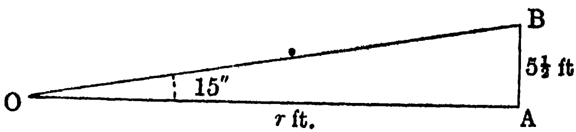
or, $$\frac{2\alpha}{\alpha + 3\beta} = 1, \text{ whence } \alpha = 3\beta.$$

$\therefore$ using (1), $3\alpha = 180$, or, $\alpha = 60$ and $\beta = \frac{\alpha}{3} = 20$.

Thus the angles are

$$60^\circ,\ 80^\circ,\ 100^\circ \text{ and } 120^\circ.$$

Ex. 5. *At what distance does a man, $5\frac{1}{2}$ ft. in height, subtend an angle of* $15''$?



AB being the man subtending an angle $15''$ at O, let OA be r ft.

As the angle AOB is very small, so that AB is very small compared to AO, we may assume the small length AB to be practically a small arc of a circle whose centre is O. Now the measure of an angle in radians is the ratio of the arc which subtends it at the centre to the radius.

[See *Art. 5*]

$$\therefore \quad \frac{15}{60 \times 60} \times \frac{\pi}{180} = \frac{5\frac{1}{2}}{r},$$

$$\text{or,} \quad r = \frac{11}{2} \times \frac{180 \times 60 \times 60}{15 \times \pi} \text{ ft.}$$

$$= \frac{11}{2} \times \frac{180 \times 60 \times 60 \times 7}{15 \times 22} \times \frac{1}{3 \times 1760} \text{ miles approx.}$$

$$= 14{\cdot}32 \text{ miles nearly.}$$

Examples I

1. Indicate the final position of a revolving line which has traced out the angle

(i) 1122° ; (ii) $-810^\circ\, 29'$;

(iii) $-617^\circ\, 51'\, 5''$; (iv) $\frac{18\pi}{5}$ radians.

2. Express (i) $55^\circ\ 12'\ 36''$ in centesimal measure ;
(ii) $195^g\ 35^`\ 24^{``}$ in degrees, minutes, and secs.

3. How many radians are there in (i) $50^g\ 75^`\ 50^{``}$; (ii) $18^\circ\ 33'\ 45''$?

4. Express in each system of angular measurement the angle between the minute-hand and the hour-hand of a clock at quarter to twelve.

5. If x^g be taken as the unit angle, and the angles, 600° and 16^c expressed in that unit be α and β respectively, find the relation between α and β.

6. The difference of two angles is 1° ; the circular measure of their sum is 1 ; find the circular measure of the smaller angle.

7. Two angles are in the ratio 2 : 3, and the difference of their measure in grades and in degrees respectively is $2\frac{1}{3}$; find the angles in degrees.

8. An angle is the excess of $D^\circ\ M'$ over $G^g\ m^`$. Find the ratio of this angle to a right angle.

9. The circular measure of a certain angle is equal to the ratio of the number of degrees in it to the number of centesimal minutes ; find the magnitude of the angle in degrees.

10. With two units of angular measurement differing by 10°, the measure of an angle are as 3 : 2 ; determine the units.

11. If an angle standing upon an arc of length 'l' at the centre of a circle of radius 'r' be taken as unit, and three angles D°, G^g, and C circular units expressed in that unit be x, y, z respectively, show that

$$x : y : z = \frac{D\pi}{18} : \frac{G\pi}{20} : 10C.$$

12. Three angles are in G.P. The number of grades in the greatest angle is to the number of circular units in the least as 800 to π, and the sum of the three angles is 126°. Find the angles in grades.

13. Divide 54° in three parts, such that the circular measure of the first exceeds that of the second by $\frac{\pi}{10}$, and the sum of the second and third is 30 grades.

14. Find at what times between 7 and 8 o'clock the angle between the two hands of a clock is (1) 60^g, (ii) 155°.

15. The angles of a triangle are in A.P., and the number of radians in the greatest is to the number of grades in the least as $\pi : 40$. Find the angles in degrees.

16. In each of two triangles the angles are in G.P.; the least angle of one of them is three times the least angle in the other, and the sum of the greatest angles is 240°. Find the circular measure of the angles.

17. One angle of a quadrilateral is $\frac{2}{3}$ of another and the two other angles are $66\frac{2}{3}$ grades and $\frac{3\pi}{4}$ radians. Express the angles in degrees.

18. The angles of a polygon (which has no reflex angle) are in A. P. The least angle is $\frac{2\pi}{3}$ radians and the common difference is 5°. Find the number of sides.

19. The number of sides of two regular polygons are as $m : n$, and the number of degrees in an angle of the first is to the number of grades in an angle of the second as $p : q$. Determine the number of sides in each polygon.

20. An arc of 50^g in one circle equals one of 60° in another; find the radian-measure of an angle subtended at the centre of the first circle by an arc equal to the radius of the second.

21. Two regular figures are such that the number of degrees in an angle of one is to the number of degrees in an angle of the other as the number of sides in the first is to the number of sides in the second. The sum of the number of sides of the two figures being 9, determine the number of sides of each.

22. The wheel of a railway carriage is 4 ft. in diameter and makes 6 revolutions in a second; how fast is the train going?

23. The earth revolves round the sun in a circular orbit of radius 92700000 miles once a year. Find its velocity in miles per hour. If the apparent angular diameter of the sun observed from the earth be $32'$, find also the linear radius of the sun.

24. A tower subtends an angle of $10'$ when the observer is at a distance of 6 miles; find its height.

25. Find the radius of the earth, if an angle of 1° is subtended at its centre by an arc joining two places on it distant 69·1 miles.

26. A horse is tied to a post by a rope 27 feet long. If the horse moves along the circumference of a circle always keeping the rope tight, find how far the horse will have gone when the rope has traced out an angle of 70°. ($\pi = \frac{22}{7}$)

27. A man running along a circular track at the rate of 10 miles per hour, traverses in 36 seconds, an arc which subtends 56° at the centre. Find the diameter of the circle. ($\pi = \frac{22}{7}$)

28. An arc of 30° in one circle is double an arc in a second circle the radius of which is three times the radius of the first. Show that the arc of the second circle subtends 5° at its centre.

CHAPTER II

TRIGONOMETRICAL RATIOS

7. Trigonometrical ratios defined.

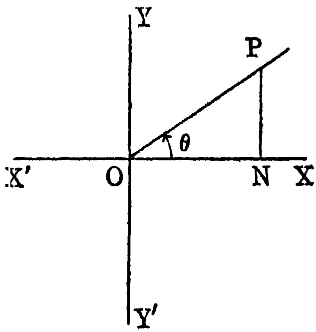


Fig. 1

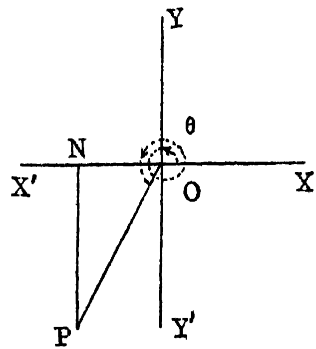


Fig. 2

Let θ be the measure of an angle XOP which may be supposed to be traced out by a revolving line starting from the initial position OX. From any point P on its other arm, draw a perpendicular PN on OX (produced if necessary, as in the second figure). A right-angled triangle is thereby formed. The trigonometrical ratios of the angle θ are defined as follows :—

Sine of the angle θ, written as $\sin\theta = \dfrac{PN}{OP}$

$$i.e., \ \frac{\text{opposite side}}{\text{hypotenuse}}$$

Cosine of θ, written as $\cos\theta = \dfrac{ON}{OP}$

$$i.e., \ \frac{\text{adjacent side}}{\text{hypotenuse}}$$

Tangent of θ, written as **tan** $\theta = \frac{PN}{ON}$

$$i.e., \frac{\textit{opposite side}}{\textit{adjacent side}}$$

Cosecant of θ, written as **cosec** $\theta = \frac{OP}{PN}$

$$i.e., \frac{\textit{hypotenuse}}{\textit{opposite side}}$$

Secant of θ, written as **sec** $\theta = \frac{OP}{ON}$

$$i.e., \frac{\textit{hypotenuse}}{\textit{adjacent side}}$$

Cotangent of θ, written as **cot** $\theta = \frac{ON}{PN}$

$$i.e., \frac{\textit{adjacent side}}{\textit{opposite side}}$$

In addition to these, we define two less important ratios of the angle θ which are sometimes used, as following :—

Versed sine of angle θ, written as **vers** $\theta = 1 - \cos\theta$

Coversed sine of angle θ, written as **covers** $\theta = 1 - \sin\theta$

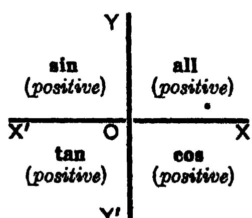
8. Signs of Trigonometrical ratios.

XOP being any angle, traced out by a revolving line which starts from OX, it has already been mentioned in the last Chapter that the plane may be divided into four quadrants by the two perpendicular lines XOX' and YOY'.

It is conventional, as in graphs, to consider distances measured along OX and OY as positive, and along OX' and OY' as negative. *The distance measured along* OP, the final position of the revolving line corresponding to the angle XOP, in whichever quadrant it may lie, *is however always considered positive.*

With this convention, if OP lies in the first quadrant as in Fig. (i) of the last article, the sides PN, ON and OP of the right-angled triangle OPN are all positive.* Hence all the trigonometrical ratios are positive. If OP lies in the third quadrant as in Fig. (ii), ON and PN are both negative, but OP is positive. Hence, from the definition of the Trigonometrical ratios, sin XOP $\left(=\frac{PN}{OP}\right)$ is negative, cos XOP $\left(=\frac{ON}{OP}\right)$ is negative, tan XOP $\left(=\frac{PN}{ON}=\frac{\text{negative quantity}}{\text{negative quantity}}\right)$ is positive etc.

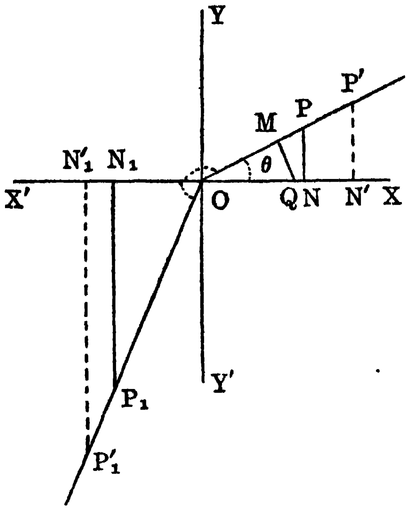
In this way, according to the final position of the revolving line (starting position being OX), we can determine the signs of the Trigonometrical ratios of the angle XOP whether this angle traced out is positive or negative. If OP is in the *first quadrant, the ratios are all positive.* If OP falls in the *second quadrant, sine and cosecant* (which is evidently the reciprocal of sine), *are positive*; all the other ratios are negative. If OP be in the *third quadrant, tangent and cotangent* (which are reciprocals to each other) *are positive*; all the others are negative. In the *fourth quadrant, cosine and secant are positive*, others are negative. A symbolical figure will help the memory in this case, namely that according to the position of OP,



The positiveness of sine, cosine and tangent also implies the positiveness of their reciprocals namely, cosecant, secant and cotangent respectively.

9. Constancy of Trigonometrical ratios.

So long as an angle remains the same, its Trigonometrical ratios are unique.



Let XOP $(=\theta)$ be any angle, and let PN and $P'N'$ be drawn perpendiculars upon OX from any two points P and P' on OP. The two right-angled triangles OPN and $OP'N'$ are similar. Hence, sin θ, whether we take it as $\dfrac{PN}{OP}$ or $\dfrac{P'N'}{OP'}$ is the same. If the angle be XOP_1, when OP_1 is not in the first quadrant, the right-angled triangles P_1N_1O and $P'_1N'_1O$ are not only similar but also have their corresponding sides of the same sign. Hence, the Trigonometrical ratios of the angle XOP_1, whether defined from the triangle P_1N_1O or from $P'_1N'_1O$ are the ---- in magnitude as well as in sign. Thus for any given angle, the Trigonometrical ratios are unique.

Note. In case of a *positive acute angle* like XOP, we might take any point Q on OX as well, and draw QM perpendicular upon OP, and define sin XOP to be $\frac{\text{opposite side}}{\text{hypotenuse}}$ *i.e.*, $\frac{QM}{OQ}$, cos XOP to be $\frac{OM}{OQ}$ etc. Now the two triangles QOM and PON are easily seen to be similar and both have their sides all positive; so that $\frac{QM}{OQ}=\frac{PN}{OP}$, $\frac{OM}{OQ}=\frac{ON}{OP}$ etc. Hence the Trigonometrical ratios of the angle XOP, even if defined from triangle QOM, will have the same values.

It may also be noted that *for angles of any magnitude, positive or negative*, any of the two arms may be supposed to be coincident with OX, and then the magnitude and sign of the angle will fix up the position of the other arm, and thereby will make the Trigonometrical ratios unique.

10. Fundamental relations between the Trigonometrical ratios of any angle.

From the very definitions given in Art. 7 of the Trigonometrical ratios of any angle XOP $(=\theta)$ of whatever magnitude and sign, we at once derive the following relations :

$$\text{cosec}\,\theta=\frac{1}{\sin\theta}$$

$$\sec\theta=\frac{1}{\cos\theta}$$

$$\cot\theta=\frac{1}{\tan\theta}$$

and since $\sin\theta=\frac{PN}{OP}$, $\cos\theta=\frac{ON}{OP}$, $\tan\theta=\frac{PN}{ON}$, $\cot\theta=\frac{ON}{PN}$,

we get $$\tan\theta=\frac{\sin\theta}{\cos\theta}$$

$$\cot\theta=\frac{\cos\theta}{\sin\theta}$$

Again, since in the right-angled triangle OPN,

$$OP^2 = PN^2 + ON^2,$$

dividing by OP^2, ON^2 and PN^2 respectively, we get

$$\left(\frac{PN}{OP}\right)^2 + \left(\frac{ON}{OP}\right)^2 = 1 \quad \cdots \quad \cdots \quad \text{(i)}$$

$$\left(\frac{OP}{ON}\right)^2 = \left(\frac{PN}{ON}\right)^2 + 1 \quad \cdots \quad \cdots \quad \text{(ii)}$$

$$\left(\frac{OP}{PN}\right)^2 = 1 + \left(\frac{ON}{PN}\right)^2 \quad \cdots \quad \cdots \quad \text{(iii)}$$

From the definition of the Trigonometrial ratios, (i) gives

$$(\sin \theta)^2 + (\cos \theta)^2 = 1.$$

Now it is usual to write $(\sin \theta)^2$ in the form $\sin^2\theta$ and so for other ratios. The relation then reduces to the form

$$\mathbf{\sin^2\theta + \cos^2\theta = 1.}$$

Similarly, (ii) and (iii) give respectively,

$$\mathbf{\sec^2\theta = 1 + \tan^2\theta}$$

$$\mathbf{\operatorname{cosec}^2\theta = 1 + \cot^2\theta.}$$

These formulæ are also used in the forms

$$\sin^2\theta = 1 - \cos^2\theta, \ \cos^2\theta = 1 - \sin^2\theta,$$

$$\sec^2\theta - \tan^2\theta = 1, \ \tan^2\theta = \sec^2\theta - 1, \text{ etc.}$$

Note. The fundamental formulæ derived in this article are very important, and are true for all values of θ whatever its magnitude and sign may be. For example, if we take $\frac{\theta}{2}$ in place of θ, we are simply taking a different angle for which the same relations are true, so that $\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1$, etc.

11. Conversions of Trigonometrical ratios.

With the help of the formulæ of the previous article, we can express any Trigonometrical ratio of an angle in terms of any other ratio for the same angle ; hence if the value of any Trigonometrical ratio of an angle be given, we can find the value of any other ratio.

Ex. 1. *Express sin θ in terms of cot θ.*

From the formulæ $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

and $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$,

we get $\sin\theta = \dfrac{1}{\operatorname{cosec}\theta} = \dfrac{1}{\pm\sqrt{1+\cot^2\theta}}$.

Ex. 2. *Express cosec θ in terms of sec θ.*

$$\operatorname{cosec}\theta = \pm\sqrt{1+\cot^2\theta} = \pm\sqrt{1+\frac{1}{\tan^2\theta}}$$

$$= \pm\sqrt{\frac{\tan^2\theta+1}{\tan^2\theta}} = \pm\sqrt{\frac{\sec^2\theta}{\sec^2\theta-1}} = \frac{\pm\sec\theta}{\sqrt{\sec^2\theta-1}}.$$

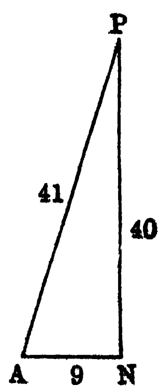
Ex. 3. *If cos* $A = \frac{12}{13}$, *find tan A.*

We have $\tan A = \dfrac{\sin A}{\cos A} = \dfrac{\pm\sqrt{1-\cos^2 A}}{\cos A}$

$$= \frac{\pm\sqrt{1-\frac{144}{169}}}{\frac{12}{13}} = \frac{\pm\frac{5}{13}}{\frac{12}{13}} = \pm\frac{5}{12}.$$

A more practical method in such cases is however to construct a right-angled triangle with the numerator and denominator as the two suitable sides, as shown below.

Ex. 4. *If sec* $A = \frac{41}{9}$, *find cot A.*



Let APN be a triangle right-angled at N in which the hypotenuse $AP = 41$, $AN = 9$,

so that $\sec NAP = \dfrac{AP}{AN} = \dfrac{41}{9}$.

Thus $\angle NAP = A$.

Now $PN^2 = AP^2 - AN^2 = 41^2 - 9^2$

$= 40^2$,

so that $PN = \pm 40$.

$\therefore$ $\cot A = \cot NAP = \dfrac{AN}{PN} = \pm\dfrac{9}{40}$.

12. Restrictions on the magnitudes of Trigonometrical ratios.

From the relation $\sin^2\theta + \cos^2\theta = 1$, since $\sin^2\theta$ and $\cos^2\theta$ being square quantities are both positive, it is evident that neither $\sin^2\theta$ nor $\cos^2\theta$ can exceed 1, for if $\sin^2\theta$, for example, be greater than 1, $\cos^2\theta$ (which is a square quantity) becomes negative, which is impossible. Thus *$\sin\theta$ as well as $\cos\theta$ must have numerical values not exceeding* 1; in other words, both $\sin\theta$ and $\cos\theta$ must lie between $+1$ and -1 whatever the magnitude of θ may be. Any value numerically greater than 1, like -2 or $+3{\cdot}1$ must be impossible for $\sin\theta$ or $\cos\theta$ so long θ is real.

$\sec\theta$ and *$\operatorname{cosec}\theta$* therefore, being reciprocals of $\cos\theta$ and $\sin\theta$ respectively, *can never be numerically less than* 1.

$\tan\theta$ and *$\cot\theta$* however, *can have any numerical value* greater than 1 or less than 1 according to the value of θ.

13. A few examples on the applications of the fundamental formulæ are given below.

Ex. 1. *Prove that* $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta.$

[*C. U. 1937*]

$$\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}} = \sqrt{\frac{(1+\cos\theta)^2}{\sin^2\theta}}$$

$$= \frac{1+\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \operatorname{cosec}\theta + \cot\theta.$$

Ex. 2. *Prove that*

$$\frac{1}{\sec A + \tan A} - \frac{1}{\cos A} = \frac{1}{\cos A} - \frac{1}{\sec A - \tan A}.$$

We have $\dfrac{1}{\sec A + \tan A} + \dfrac{1}{\sec A - \tan A}$

$$= \frac{\sec A - \tan A + \sec A + \tan A}{(\sec A + \tan A)(\sec A - \tan A)} = \frac{2\sec A}{\sec^2 A - \tan^2 A}$$

$$= 2\sec A = \frac{2}{\cos A} = \frac{1}{\cos A} + \frac{1}{\cos A}.$$

Hence, by transposition,

$$\frac{1}{\sec A+\cos A}-\frac{1}{\cos A}=\frac{1}{\cos A}-\frac{1}{\sec A-\tan A}.$$

Ex. 3. *Prove that* $\dfrac{1+2\sin\theta\cos\theta}{(\sin\theta+\cos\theta)(\cot\theta+\tan\theta)}$
$=\sin\theta\cos\theta\,(\sin\theta+\cos\theta).$

We have $\dfrac{1+2\sin\theta\cos\theta}{(\sin\theta+\cos\theta)(\cot\theta+\tan\theta)}$

$$=\frac{(\sin^2\theta+\cos^2\theta)+2\sin\theta\cos\theta}{(\sin\theta+\cos\theta)\left(\dfrac{\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{\cos\theta}\right)}$$

$$=\frac{(\sin\theta+\cos\theta)^2}{(\sin\theta+\cos\theta)\left(\dfrac{\cos^2\theta+\sin^2\theta}{\sin\theta\cos\theta}\right)}$$

$$=\frac{(\sin\theta+\cos\theta)\sin\theta\cos\theta}{1}$$

$$=\sin\theta\cos\theta\,(\sin\theta+\cos\theta).$$

Ex. 4. *If* $15\sin^2\theta+2\cos\theta=7$, *find* $\tan\theta$.

Here $15(1-\cos^2\theta)+2\cos\theta=7,$

whence $15\cos^2\theta-2\cos\theta-8=0,$

or, $(5\cos\theta-4)(3\cos\theta+2)=0$; $\therefore\ \cos\theta=\frac{4}{5}$, or, $-\frac{2}{3}$.

Case (i) when $\cos\theta=\frac{4}{5}$,

$\sin^2\theta=1-\cos^2\theta=1-\frac{16}{25}=\frac{9}{25}$. $\therefore\ \sin\theta=\pm\frac{3}{5}$,

and so $\tan\theta=\dfrac{\sin\theta}{\cos\theta}=\pm\dfrac{3}{4}$.

Case (ii) when $\cos\theta=-\frac{2}{3}$,

$\sin^2\theta=1-\cos^2\theta=1-\frac{4}{9}=\frac{5}{9}$. $\therefore\ \sin\theta=\pm\dfrac{\sqrt{5}}{3}$.

$\therefore\ \tan\theta=\dfrac{\sin\theta}{\cos\theta}=\pm\dfrac{\sqrt{5}}{2}$.

Examples II

Prove the following identities (*Ex. 1 to 24*) :—

1. $\dfrac{\sin A + \cos A}{\sec A + \operatorname{cosec} A} = \sin A \cos A.$

2. $\cot\theta + \tan\theta = \sec\theta \operatorname{cosec}\theta.$

3. $\dfrac{1}{1+\tan A} = \dfrac{\cot A}{1+\cot A}.$

4. $\operatorname{cosec}^6 A - \cot^6 A = 1 + 3 \operatorname{cosec}^2 A \cot^2 A.$

5. $\cos^6 A + \sin^6 A = 1 - 3 \sin^2 A \cos^2 A.$

6. $\dfrac{1}{\cos^2 A} - \dfrac{1}{\operatorname{cosec}^2 A - 1} = 1.$

7. $\cos A + \tan A \sin A = \sec A.$

8. $\sec^4 A + \tan^4 A = 1 + 2 \sec^2 A \tan^2 A.$

9. $\dfrac{1 + 3\cos\theta - 4\cos^3\theta}{1-\cos\theta} = (1 + 2\cos\theta)^2.$

10. $(\cot\theta + \operatorname{cosec}\theta)^2 = \dfrac{1+\cos\theta}{1-\cos\theta}.$

11. $\dfrac{1+\tan^2\theta}{1+\cot^2\theta} = \left(\dfrac{1-\tan\theta}{1-\cot\theta}\right)^2.$

12. $\dfrac{\tan^2\alpha - \cot^2\alpha}{1+\cot^2\alpha} = \dfrac{\sin^2\alpha - \cos^2\alpha}{\cos^2\alpha}.$

13. $1 + \tan\theta + \sec\theta = \dfrac{2}{1 + \cot\theta - \operatorname{cosec}\theta}$

14. $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \dfrac{1+\sin\theta}{\cos\theta}$

15. $\dfrac{\sin A - 2\sin^3 A}{2\cos^3 A - \cos A} = \tan A.$

16. $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} - \sec\theta = \sec\theta - \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}.$

17. $\dfrac{\operatorname{cosec} A + \cot A}{\operatorname{cosec} A - \cot A} = \dfrac{\sin^2 A}{(1 - \cos A)^2}$.

18. $(1 + \sin A + \cos A)^2 = 2(1 + \sin A)(1 + \cos A)$.

19. $\dfrac{\sec\theta + \tan\theta}{\operatorname{cosec}\theta + \cot\theta} - \dfrac{\sec\theta - \tan\theta}{\operatorname{cosec}\theta - \cot\theta} = 2\,(\sec\theta - \operatorname{cosec}\theta)$.

20. $\dfrac{1}{1 + \sin^2\theta} + \dfrac{1}{1 + \operatorname{cosec}^2\theta} = 1$.

21. $\dfrac{\sin^3\alpha + \cos^3\alpha}{\sin\alpha + \cos\alpha} + \dfrac{\sin^3\alpha - \cos^3\alpha}{\sin\alpha - \cos\alpha} = 2$.

22. $\dfrac{\tan\theta}{\sec\theta - 1} - \dfrac{\sin\theta}{1 + \cos\theta} = 2\cot\theta$.

23. $\dfrac{\cos\theta + \cos\phi}{\sin\theta - \sin\phi} = \dfrac{\sin\theta + \sin\phi}{\cos\phi - \cos\theta}$.

24. $1 + 4\operatorname{cosec}^2\theta\cot^2\theta = (\operatorname{cosec}^2\theta + \cot^2\theta)^2$.

25. Express $1 - 2\sin\theta\cos\theta$ as a perfect square.

26. Express $2\sec^2\theta - \sec^4\theta - 2\operatorname{cosec}^2\theta + \operatorname{cosec}^4\theta$ in terms of $\tan\theta$.

27. Prove that

$$(\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\alpha\cos\beta - \cos\alpha\sin\beta) = \sin^2\alpha - \sin^2\beta.$$

28. If $\sin A + \sin^2 A = 1$, then $\cos^2 A + \cos^4 A = 1$.

29. (i) If $\sin\theta - \cos\theta = 0$, prove that $\sec\theta = \pm\sqrt{2}$.

(ii) If $7\sin^2\theta + 3\cos^2\theta = 4$, show that $\tan\theta = \pm\dfrac{1}{\sqrt{3}}$.

(iii) If $3\sin\theta + 4\cos\theta = 5$, show that $\sin\theta = \frac{3}{5}$.

30. If $\tan\theta + \sec\theta = x$, show that $\sin\theta = \dfrac{x^2 - 1}{x^2 + 1}$.

31. If $\tan\theta = \dfrac{a}{b}$, find the value of $\dfrac{a\sin\theta - b\cos\theta}{a\sin\theta + b\cos\theta}$.

32. If $1+4x^2=4x \sec A$, prove that

$$\sec A+\tan A=2x \text{ or } 1/2x.$$

33. Express $\sin \alpha$ in terms of $\sec \alpha$, and $\sec \theta$ in terms of $\cot \theta$.

34. Given $\sin \theta=\frac{3}{5}$, $\cos \phi=\frac{12}{13}$, where θ and ϕ are acute angles, find the value of $\dfrac{\tan \theta-\tan \phi}{1+\tan \theta \tan \phi}$.

35. If $\cos \alpha+\sin \alpha=\sqrt{2} \cos \alpha$, prove that

$$\cos \alpha-\sin \alpha=\sqrt{2} \sin \alpha.$$

36. If $\tan A=\dfrac{1}{\sqrt{3}}$, find $\dfrac{\operatorname{cosec}^2 A-\sec^2 A}{\operatorname{cosec}^2 A+\sec^2 A}$.

37. If $1+\sin^2 A=3 \sin A \cos A$, find $\tan A$.

38. If $\tan \theta+\sin \theta=m$, $\tan \theta-\sin \theta=n$, prove that

$$m^2-n^2=4\sqrt{mn}.$$

39. If $(a^2-b^2) \sin \theta+2ab \cos \theta=a^2+b^2$, find $\tan \theta$ and $\operatorname{cosec} \theta$.

40. If $\tan \theta=\dfrac{\sin \alpha-\cos \alpha}{\sin \alpha+\cos \alpha}$, prove that

$$\sqrt{2} \cos \theta=\sin \alpha+\cos \alpha.$$

41. Given $\tan^2 \theta=1-e^2$, show that

$$\sec \theta+\tan^3 \theta \operatorname{cosec} \theta=(2-e^2)^{\frac{3}{2}}.$$

42. If x and y are two unequal real quantities, show that the equations (i) $\sin^2 \theta=\dfrac{(x+y)^2}{4xy}$ and (ii) $\cos \theta=x+\dfrac{1}{x}$ are both impossible.

43. Eliminate θ between

(i) $x=a \cos \theta$, $y=b \sin \theta$.

(ii) $x=c(\sec \theta+\tan \theta)$, $y=c(\sec \theta-\tan \theta)$.

(iii) $a \cos \theta+b \sin \theta+c=0$, $a' \cos \theta+b' \sin \theta+c'=0$.

(iv) $a \tan^2 \theta+b \tan \theta+c=a' \cot^2 \theta+b' \cot \theta+c'=0$.

Examples II(A)

Prove the following identities (*Ex. 1 to 18*) :—

1. $\dfrac{\tan^3 \alpha}{1+\tan^2 \alpha}+\dfrac{\cot^3 \alpha}{1+\cot^2 \alpha}=\dfrac{1-2 \sin^2 \alpha \cos^2 \alpha}{\sin \alpha \cos \alpha}.$

2. $(\tan \theta+\cot \theta+\sec \theta)(\tan \theta+\cot \theta-\sec \theta)=\operatorname{cosec}^2 \theta.$

3. $\sin \theta\,(1+\tan \theta)+\cos \theta\,(1+\cot \theta)=\sec \theta+\operatorname{cosec} \theta.$

[*C. U. 1935*]

4. $(1+\sin \alpha-\cos \alpha)^2+(1-\sin \alpha+\cos \alpha)^2$
$$=4(1-\sin \alpha \cos \alpha).$$

5. $\sin^6 \alpha+\sin^4 \alpha \cos^2 \alpha-\sin^2 \alpha \cos^4 \alpha-\cos^6 \alpha$
$$=\sin^2 \alpha-\cos^2 \alpha.$$

6. $3\,(\sin \theta+\cos \theta)-2\,(\sin^3 \theta+\cos^3 \theta)=(\sin \theta+\cos \theta)^3.$

7. $\dfrac{1}{\operatorname{cosec} \theta-\cot \theta}-\dfrac{1}{\sin \theta}=\dfrac{1}{\sin \theta}-\dfrac{1}{\operatorname{cosec} \theta+\cot \theta}.$

8. $\dfrac{\cos x}{\sin x+\cos y}+\dfrac{\cos y}{\sin y-\cos x}=\dfrac{\cos x}{\sin x-\cos y}+\dfrac{\cos y}{\sin y+\cos x}.$

9. $(\sin \theta+\operatorname{cosec} \theta)^2+(\cos \theta+\sec \theta)^2=\tan^2 \theta+\cot^2 \theta+7.$

10. $(\sec \theta-\cos \theta)(\operatorname{cosec} \theta-\sin \theta)(\tan \theta+\cot \theta)=1.$

11. $\dfrac{1+(\operatorname{cosec} x \tan y)^2}{1+(\operatorname{cosec} z \tan y)^2}=\dfrac{1+(\cot x \sin y)^2}{1+(\cot z \sin y)^2}.$

12. $\sec^3 \alpha \operatorname{cosec}^3 \alpha-3 \sec \alpha \operatorname{cosec} \alpha=\tan^3 \alpha+\cot^3 \alpha.$

13. $\sin^6 A-\cos^6 A=(\sin A+\cos A)(\sin A-\cos A)$
$$\times(1+\sin A \cos A)(1-\sin A \cos A).$$

14. $\dfrac{\tan \alpha}{(1+\tan^2 \alpha)^2}+\dfrac{\cot \alpha}{(1+\cot^2 \alpha)^2}=\sin \alpha \cos \alpha.$

15. $\sin^2 \theta \tan \theta-\cos^2 \theta \cot \theta+\sec \theta \operatorname{cosec} \theta=2 \tan \theta.$

16. $\dfrac{\cos^2 A-\sin^2 A}{\sin A \cos^2 A-\cos A \sin^2 A}=\operatorname{cosec} A+\sec A.$

17. $\dfrac{\tan^2 A+\cot^2 A}{\tan^2 A-\cot^2 A}=\dfrac{\sin^4 A+\cos^4 A}{\sin^2 A-\cos^2 A}.$

18. $(\sin\alpha\cos\beta-\cos\alpha\sin\beta)^2$
$$+(\cos\alpha\cos\beta+\sin\alpha\sin\beta)^2=1.$$

19. If $\cos^2A-\sin^2A=\tan^2B$,
$$\text{then } \cos^2B-\sin^2B=\tan^2A.$$

20. If $\sin^4x+\sin^2x=1$, then $\tan^4x-\tan^2x=1$.

21. Show that the difference between $3\sin^4\theta-2\sin^6\theta$ and $2\cos^6\theta-3\cos^4\theta$ is the same for all values of θ.

22. If $x=\dfrac{1+\sin\theta}{\cos\theta}$, show that $\dfrac{1}{x}=\dfrac{1-\sin\theta}{\cos\theta}$.

23. If $\tan^2A=1+2\tan^2B$, show that $\cos^2B=2\cos^2A$.

24. If $\sin\alpha+\cos\alpha=1$, then $\sin\alpha-\cos\alpha=\pm 1$.

25. If $a\cos\theta-b\sin\theta=c$, then show that
$$a\sin\theta+b\cos\theta=\pm\sqrt{a^2+b^2-c^2}.$$

26. If $(1+\sin x)(1+\sin y)(1+\sin z)$
$$=(1-\sin x)(1-\sin y)(1-\sin z),$$
prove that each is equal to $\pm\cos x\cos y\cos z$.

27. If $x\sin^3\alpha+y\cos^3\alpha=\sin\alpha\cos\alpha$, and
$x\sin\alpha-y\cos\alpha=0$, then $x^2+y^2=1$. [*C. U. 1937*]

28. If $\sin A=\dfrac{\sin x+\sin y}{1+\sin x\sin y}$, show that
$$\cos A=\pm\frac{\cos x\cos y}{1+\sin x\sin y}.$$

29. (i) If $\sin\alpha+\operatorname{cosec}\alpha=2$,
$$\text{then } \sin^n\alpha+\operatorname{cosec}^n\alpha=2.$$

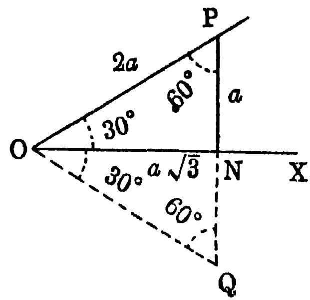
(ii) If $\sec\alpha=\sec\beta\sec\gamma+\tan\beta\tan\gamma$,
$$\text{then } \sec\beta=\sec\gamma\sec\alpha\pm\tan\gamma\tan\alpha.$$

30. If $\dfrac{\cos^4x}{\cos^2y}+\dfrac{\sin^4x}{\sin^2y}=1$, then $\dfrac{\cos^4y}{\cos^2x}+\dfrac{\sin^4y}{\sin^2x}=1$.

CHAPTER III

TRIGONOMETRICAL RATIOS OF SOME STANDARD ANGLES

14. Ratios of 30°.



Let the angle XOP, which may be supposed to be traced out by a revolving line starting from OX, be 30°. Let PN be drawn perpendicular upon OX from any point P on OP. The angle OPN is then 60°.

Produce PN to Q, making $NQ = NP$. Join OQ. The triangles PON and QON are easily seen to be equal in all respects, and so $\angle OQN = \angle OPN = 60°$. Hence, the triangle OPQ is equilateral, and so $OP = PQ =$ double of PN.

Hence, in the above figure if $PN = a$, then $OP = 2a$ and so $ON = \sqrt{OP^2 - PN^2} = \sqrt{4a^2 - a^2} = \sqrt{3}a$. The sides ON PN, and OP are all positive in this case, since the angle is acute.

Hence,

$$\sin 30° = \sin PON = \frac{PN}{OP} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30° = \frac{ON}{OP} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

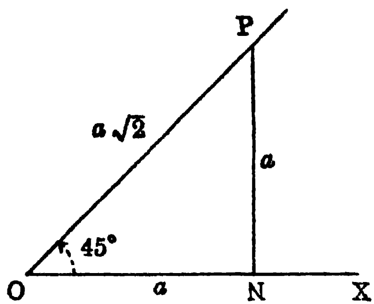
$$\tan 30^\circ = \frac{PN}{ON} = \frac{1}{\sqrt{3}}$$

$$\cot 30^\circ = \frac{ON}{PN} = \sqrt{3}$$

$$\operatorname{cosec} 30^\circ = \frac{1}{\sin 30^\circ} = 2$$

$$\sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}}.$$

15. Ratios of 45°.



Let $\angle XOP = 45^\circ$. PN is perpendicular on OX. In the right-angled triangle PON, $\angle PON = 45^\circ$.

Therefore, $\angle OPN$ is also 45° and so $ON = PN = a$ suppose. Then $OP = \sqrt{ON^2 + PN^2} = \sqrt{a^2 + a^2} = a\sqrt{2}$.

Hence,

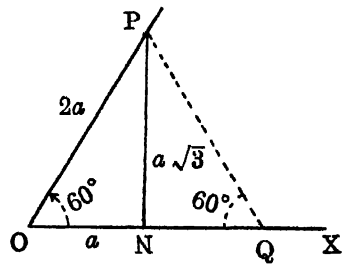
$$\sin 45^\circ = \frac{PN}{OP} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{ON}{OP} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{PN}{ON} = 1$$

$\sec 45^\circ = \operatorname{cosec} 45^\circ = \sqrt{2}$, $\cot 45^\circ = 1$.

16. Ratios of 60°.



Let $\angle XOP = 60°$. Now PN being perpendicular upon OX, along NX cut off $NQ = ON$. Join PQ. Then the two triangles OPN and QPN are easily seen to be congruent. Hence, $\angle PQN = \angle PON = 60°$. Thus the triangle POQ is equilateral, and so $OP = OQ =$ double of ON.

If $ON = a$, then $OP = 2a$ and hence $PN = \sqrt{OP^2 - ON^2}$ $= a\sqrt{3}$.

Then $$\sin 60° = \frac{PN}{OP} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{ON}{OP} = \frac{1}{2}$$

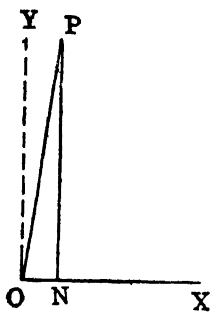
$$\tan 60° = \frac{PN}{ON} = \sqrt{3}$$

$$\cot 60° = \frac{1}{\sqrt{3}}, \ \sec 60° = 2, \operatorname{cosec} 60° = \frac{2}{\sqrt{3}}.$$

Note. It may be noted from the values of the ratios that $\sin 60° = \cos 30°$, $\cos 60° = \sin 30°$, $\tan 60° = \cot 30°$, $\cot 60° = \tan 30°$, $\sec 60° = \operatorname{cosec} 30°$, $\operatorname{cosec} 60° = \sec 30°$. It will be proved more generally, in the next chapter, that for any two complementary angles sine of one is the cosine of the other and *vice versa*, tangent of one is the co-tangent of the other, and secant of one is the cosecant of the other. The angle 45° being its own complement, therefore, it should have its sine and cosine equal to one another, as is actually seen to be the case.

17. Ratios of 90°.

Let XOP be an acute angle very nearly 90°. PN being perpendicular upon OX, ON is extremely small, and as $\angle XOP$ approaches more and more to 90°, ON becomes smaller and smaller. The length OP may however remain finite, and PN and OP will approach each other more and more closely. Ultimately when $\angle XOP$ becomes 90°, OP and PN coincide, and ON becomes zero ultimately. Hence the ratio PN/OP becomes 1 and ON/OP becomes zero.



Thus $$\sin 90° = \frac{PN}{OP} \text{ in the limit} = 1$$

$$\cos 90° = \frac{ON}{OP} \text{ in the limit} = 0$$

$$\tan 90° = \frac{PN}{ON} \text{ in the limit} = \infty\text{* (infinity)}$$

(since $ON \to 0$, whereas PN remains finite)

$$\cot 90° = \frac{\cos 90°}{\sin 90°} = \frac{0}{1} = 0$$

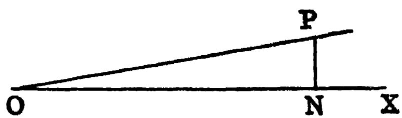
cosec $90° = 1$, sec $90° = OP/ON$ in the limit $= \infty$*.

---

* The symbol ∞ is used to denote a quantity which exceeds any positive number, however large, and does not represent a definite number.

It should be noted that in determining tan 90°, we may start with an angle XOP, slightly greater than 90°, (*i.e.*, in the second quadrant), and make it approach 90°. Then ON will be negative and $\to 0$, whereas PN is positive. Accordingly we may also write $\tan 90° = -\infty$. Thus strictly speaking, we should write $\tan 90° = \pm\infty$. Similar remarks apply for sec 90°, cot 0°, cosec 0°.

18. Ratios of 0°.



Let $\angle XOP$ be an infinitely small positive angle, and let PN be perpendicular on OX.

Then, PN is infinitely small, whereas OP is finite. Now if $\angle XOP$ be taken less and less and ultimately becomes less than any quantity we can assign, we denote it by zero, and in this case PN practically vanishes, whereas OP and ON remaining finite, coincide. Hence, the ratio PN/OP becomes ultimately zero, and ON/OP becomes 1.

Hence, $$\sin 0° = \frac{PN}{OP} \text{ in the limit} = 0$$

$$\cos 0° = \frac{ON}{OP} \text{ in the limit} = 1$$

$$\tan 0° = \frac{\sin 0°}{\cos 0°} = \frac{0}{1} = 0$$

$$\cot 0° = \frac{ON}{PN} \text{ in the limit} = \infty^*$$

$$\operatorname{cosec} 0° = \frac{OP}{PN} \text{ in the limit} = \infty^*$$

$$\sec 0° = \frac{1}{\cos 0°} = \frac{1}{1} = 1.$$

Note. Note that 0° and 90° being complementary,

$\sin 0° = \cos 90° = 0$, $\cos 0° = \sin 90° = 1$, etc.

19. As the ratios of the standard angles 0°, 30°, 45°, 60° and 90° are very often used, they should be remembered very

* See foot note of Art. 17.

carefully. The first three ratios are given in the tabulated form below. The other three are reciprocals to these.

angle	sine	cosine	tangent
0° or 0°	0	1	0
30° or $\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45° or $\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60° or $\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90° or $\frac{\pi}{2}$	1	0	$\pm\infty$

Note. The following device may be of use in remembering the *sines* and *cosines* of standard angles. The sines of the angles 0°, 30°, 45°, 60°, 90° are respectively the square roots of the fractions

$$\tfrac{0}{4},\ \tfrac{1}{4},\ \tfrac{2}{4},\ \tfrac{3}{4},\ \tfrac{4}{4}$$

and cosines of these are the square roots from right to left.

20. Examples worked out.

Ex. 1. *If* $\theta = 30^\circ$, *verify that* $\cos 2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$.

Here, $\cos 2\theta = \cos 60^\circ = \dfrac{1}{2}$. Also $\tan\theta = \tan 30^\circ = \dfrac{1}{\sqrt{3}}$.

$$\therefore \quad \frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}} = \frac{1}{2}.$$

Hence, $\cos 2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$.

Ex. 2. *Verify that*

$$\sin 30^\circ = \sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ.$$

The right-hand side, on substitution of the values,

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2} = \sin 30^\circ.$$

Hence the result.

Ex. 3. *Solve for θ, where θ is a positive acute angle, given $\operatorname{cosec}\theta \cot\theta = 2\sqrt{3}$.*

From the given equation, $\dfrac{1}{\sin\theta} \cdot \dfrac{\cos\theta}{\sin\theta} = 2\sqrt{3}$,

$$\text{or,}\quad \cos\theta = 2\sqrt{3}\sin^2\theta = 2\sqrt{3}(1-\cos^2\theta),$$

whence, $2\sqrt{3}\cos^2\theta + \cos\theta - 2\sqrt{3} = 0$

giving $\cos\theta = \dfrac{-1 \pm \sqrt{1+48}}{4\sqrt{3}} = \dfrac{-1 \pm 7}{4\sqrt{3}}$.

Since θ is a positive acute angle, $\cos\theta$ is positive, and so rejecting the negative value,

$$\cos\theta = \frac{6}{4\sqrt{3}} = \frac{\sqrt{3}}{2} = \cos 30^\circ. \quad \therefore \quad \theta = 30^\circ \text{ i.e., } \frac{\pi}{6}.$$

Examples III

Verify the results (*Ex. 1 to 6*) :—

1. $1 - 2\sin^2 30^\circ = 2\cos^2 30^\circ - 1 = \cos 60^\circ$.

2. $\dfrac{2\tan 30^\circ}{1-\tan^2 30^\circ} = \sqrt{3}$.

3. $\cos\dfrac{\pi}{3}\cos\dfrac{\pi}{4} + \sin\dfrac{\pi}{3}\sin\dfrac{\pi}{4} = \sin\dfrac{\pi}{4}\cos\dfrac{\pi}{6} + \cos\dfrac{\pi}{4}\sin\dfrac{\pi}{6}$.

4. (i) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

(ii) $\cos A = \cos^2 B - \sin^2 B$, where $A = 60^\circ$, $B = 30^\circ$.

5. $\sin 3A = 3 \sin A - 4 \sin^3 A$, where $A = \frac{\pi}{6}$.

6. $\operatorname{cosec}^2 45^\circ \sec^2 30^\circ (\sin^3 30^\circ + 4 \cot^2 45^\circ - \sec^2 60^\circ) = \frac{1}{3}$.

7. If $\tan^2 \frac{\pi}{4} - \cos^2 \frac{\pi}{3} = x \sin \frac{\pi}{4} \cos \frac{\pi}{4} \tan \frac{\pi}{3}$, find x.

8. If θ be a positive acute angle, find θ, when

(i) $2 \sin^2 \theta = 3 \cos \theta$.

(ii) $\tan \theta + \cot \theta = 2$.

(iii) $\operatorname{cosec}^2 \theta + 5 = 3 \sqrt{3} \cot \theta$.

(iv) $\sin \theta + \cos \theta = \sqrt{2}$.

(v) $2 (\cos^2 \theta - \sin^2 \theta) = 1$.

(vi) $6 \sin^2 \theta - 11 \sin \theta + 4 = 0$.

(vii) $\dfrac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta} = \dfrac{1 - \sqrt{3}}{1 + \sqrt{3}}$.

9. Given θ and ϕ to be positive acute angles, and $\tan (\theta + \phi) = \sqrt{3}$, $\tan (\theta - \phi) = 1$, determine θ and ϕ.

10. Find α and β (α and β being positive acute angles), if

$$\sin (2\alpha - \beta) = 1,$$

and $$\cos (\alpha + \beta) = \tfrac{1}{2}.$$

11. Find A, B, C (A, B, C being positive acute angles), if

$$\sin (B + C - A) = 1,$$
$$\cos (C + A - B) = 1,$$

and $$\tan (A + B - C) = 1.$$

12. Find the numerical values of :—

(i) $\cot^2 \frac{\pi}{6} - 2 \cos^2 \frac{\pi}{3} - \frac{3}{4} \sec^2 \frac{\pi}{4} - 4 \sec^2 \frac{\pi}{6}$.

(ii) $3 \tan^2 45^\circ - \sin^2 60^\circ - \frac{1}{2} \cot^2 30^\circ + \frac{1}{8} \sec^2 45^\circ$.

CHAPTER IV

TRIGONOMETRICAL RATIOS OF ANGLES ASSOCIATED WITH A GIVEN ANGLE θ

21. Ratios of the angle $(-\theta)$ *in terms of those of* θ, θ *having any magnitude.*

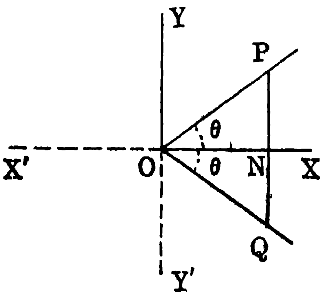


Fig. (i)

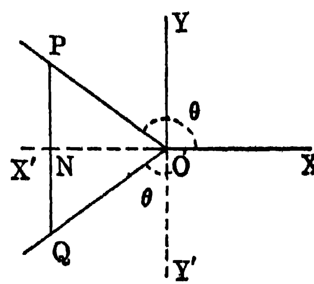


Fig. (ii)

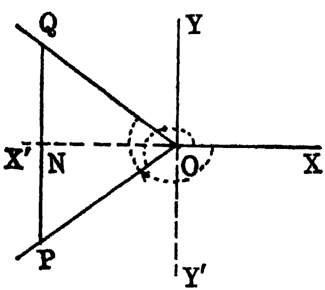


Fig. (iii)

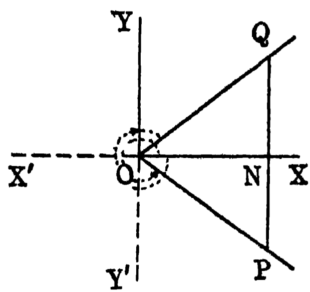


Fig. (iv)

Let the $\angle XOP$ be θ and the $\angle XOQ$ described clockwisely be $-\theta$. From any point P on OP draw PN perpendicular to OX [or OX' as in Figs. (ii) and (iii)], and produce it to meet OQ at Q say.

Now, $\angle XOP$ (measured anti-clockwisely) being equal to $\angle XOQ$ (measured clockwisely), $\angle PON = \angle QON$ in magnitude in all the figures, and therefore, the two rt.-angled triangles PON and QON are congruent. The corresponding sides are therefore equal in magnitude. Considering the signs of these sides according to the usual convention, we get in all the figures,

$$QN = -PN, \text{ and } OQ = OP$$

(both OP and OQ being always considered positive).

Hence, from definition,

$$\sin(-\theta) = \frac{QN}{OQ} = \frac{-PN}{OP} = -\sin\theta$$

$$\cos(-\theta) = \frac{ON}{OQ} = \frac{ON}{OP} = \cos\theta$$

$$\tan(-\theta) = \frac{QN}{ON} = \frac{-PN}{ON} = -\tan\theta$$

and the reciprocals of these give,

$$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta,$$

$$\sec(-\theta) = \sec\theta,$$

$$\cot(-\theta) = -\cot\theta.$$

22. Ratios of $(90° - \theta)$.

Let the $\angle XOP$ traced out by a revolving line be θ, and let another revolving line, starting from OX trace out the angle $XOY = 90°$ and then revolve back, tracing out $\angle YOQ = \theta$ in the clockwise direction, so that $\angle XOQ = 90° - \theta$.

Take two equal lengths OP and OQ along OP and OQ respectively, and draw PN and QM perpendiculars on OX.

If *OP be in the first or third quadrant* as in Fig. (i) and Fig. (iii), OQ also lies in the same quadrant. If OP lies in the second quadrant as in Fig. (ii), OQ lies in the fourth quadrant; and if OP lies in the fourth, OQ lies in the

second, as in Fig. (iv). Now, $\angle XOP$ being equal to $\angle YOQ$ in magnitude, $\angle PON = \angle OQM$, and since $OP = OQ$,

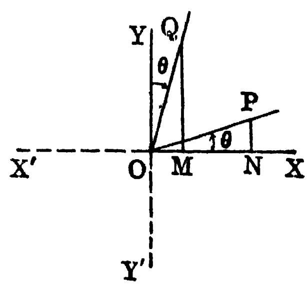


Fig. (i)

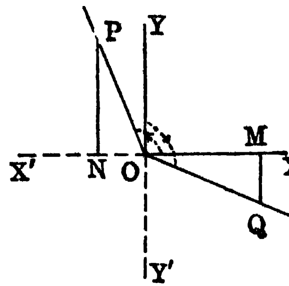


Fig. (ii)

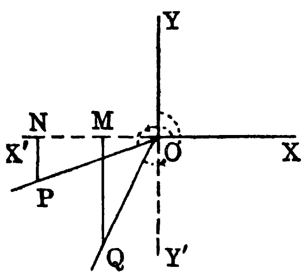


Fig. (iii)

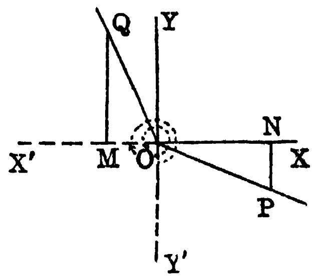


Fig. (iv)

the two rt.-angled triangles PON, OQM are congruent. The corresponding sides are therefore equal in magnitude. Considering signs as well, we get in all the figures,

$$QM = ON,\ OM = PN,\ OQ = OP.$$

Hence from definition,

$$\textbf{sin}\ (90° - \theta) = \sin \angle XOQ = \frac{QM}{OQ} = \frac{ON}{OP} = \textbf{cos}\ \theta$$

$$\textbf{cos}\ (90° - \theta) = \frac{OM}{OQ} = \frac{PN}{OP} = \textbf{sin}\ \theta$$

$$\textbf{tan}\ (90° - \theta) = \frac{QM}{OM} = \frac{ON}{PN} = \textbf{cot}\ \theta$$

The reciprocals of these are

$$\operatorname{cosec}(90^\circ-\theta)=\sec\theta,$$
$$\sec(90^\circ-\theta)=\operatorname{cosec}\theta,$$
$$\cot(90^\circ-\theta)=\tan\theta.$$

Obs. The angles $(90^\circ-\theta)$ is the complement of θ, and we derive the result that for a pair of complementary angles sine of one is the cosine of the other and *vice versa*, tangent of one is the cotangent of the other and secant of one is the cosecant of the other. This was verified in the last chapter in connection with the complementary pairs 30° and 60°, as also 0° and 90°.

23. Ratios of $(90^\circ+\theta)$.

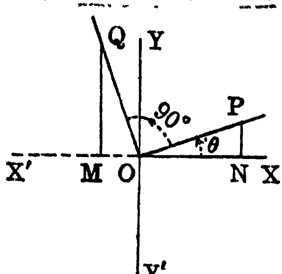


Fig. (i)

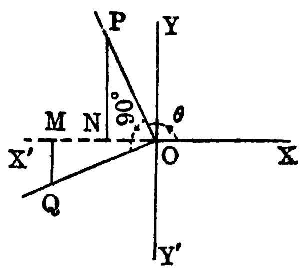


Fig. (ii)

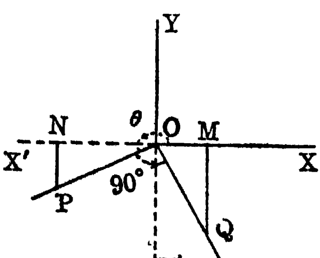


Fig. (iii)

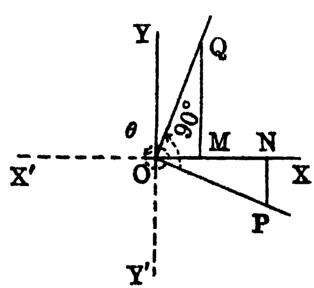


Fig. (iv)

Let a revolving line, starting from OX, trace out an $\angle XOP=\theta$, and further trace out an $\angle POQ=90^\circ$, so that $\angle XOQ=90^\circ+\theta$.

Cut off $OP = OQ$ along OP and OQ respectively and let PN, QM be perpendiculars on OX (produced where necessary).

Now, OQ being perpendicular to OP, the $\angle PON$ = the complement of $\angle QOM = \angle OQM$ in magnitude, and since $OP = OQ$, the two right-angled triangles OPN and OQM are congruent. The corresponding sides are therefore equal. Considering signs as well, we get, for all the figures,

$$QM = ON,\ OM = -PN,\ OQ = OP.$$

Hence, from definition,

$$\sin(90^\circ + \theta) = \sin \angle XOQ = \frac{QM}{OQ} = \frac{ON}{OP} = \cos\theta$$

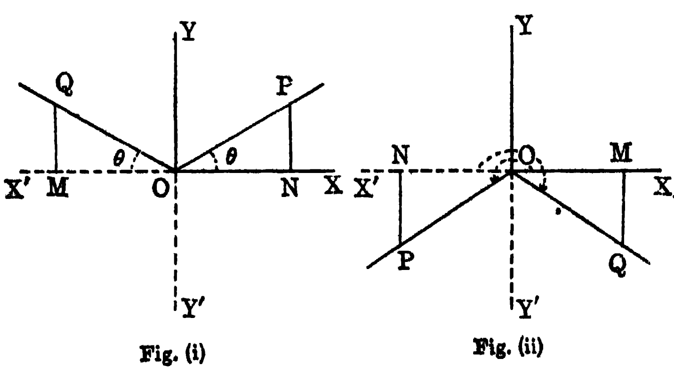
$$\cos(90^\circ + \theta) = \frac{OM}{OQ} = \frac{-PN}{OP} = -\sin\theta$$

$$\tan(90^\circ + \theta) = \frac{QM}{OM} = \frac{ON}{-PN} = -\cot\theta$$

and considering their reciprocals,

$$\operatorname{cosec}(90^\circ + \theta) = \sec\theta,$$
$$\sec(90^\circ + \theta) = -\operatorname{cosec}\theta,$$
$$\cot(90^\circ + \theta) = -\tan\theta.$$

24. Ratios of $(180^\circ - \theta)$.



Let $\angle XOP = \theta$ be traced out by a revolving line, and let another revolving line, starting from OX, trace out an angle $180°$ coming up to OX' and then revolve back and describe an angle $X'OQ = \theta$, so that $\angle XOQ = 180° - \theta$.

Two figures are given here, one with OP in the first quadrant and another with OP in the third quadrant. The two other figures may easily be drawn by the students.

Now cut off $OP = OQ$, and draw PN and QM perpendiculars on OX (or OX' as the case may be). Then $\angle PON = \angle QOM$ in magnitude, and $OP = OQ$. Hence, the right-angled triangles PON and QOM are congruent, and so have their corresponding sides equal in magnitude. Taking into consideration the signs, we get for all the figures,

$$QM = PN,\ OM = -ON,\ OQ = OP.$$

Hence, for all values of θ,

$$\sin(180° - \theta) = \sin XOQ = \frac{QM}{OQ} = \frac{PN}{OP} = \sin\theta$$

$$\cos(180° - \theta) = \frac{OM}{OP} = \frac{-ON}{OP} = -\cos\theta$$

$$\tan(180° - \theta) = \frac{QM}{OM} = \frac{PN}{-ON} = -\tan\theta$$

and so taking reciprocals,

$$\operatorname{cosec}(180° - \theta) = \operatorname{cosec}\theta,$$

$$\sec(180° - \theta) = -\sec\theta,$$

$$\cot(180° - \theta) = -\cot\theta.$$

Note. The first two formulæ may be expressed in the form "*sines of supplementary angles are equal, and cosines of supplementary angles are equal in magnitude but opposite in sign.*"

25. Ratios of $(180° + \theta)$.

Let a revolving line starting from OX, trace out an angle $XOP = \theta$, and further trace out an angle $POQ = 180°$, so that $\angle XOQ = 180° + \theta$.

OP and OQ are then in one straight line.

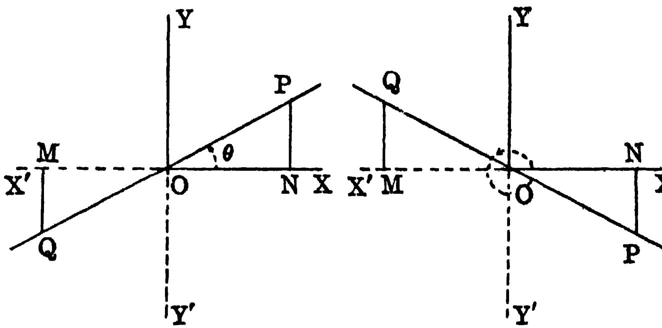


Fig. (i) Fig. (ii)

Cut off $OP = OQ$, and draw PN and QM perpendiculars on XOX'.

Two figures are given here with OP in the first and fourth quadrants, and the other two may be similarly drawn.

Now, POQ being a straight line in this case, $\angle PON = \angle QOM$ in magnitude. Also, $OP = OQ$. Hence, the right-angled triangles PON and QOM are congruent and have their corresponding sides equal in magnitude. Considering signs, we get in all cases,

$$QM = -PN,\ OM = -ON,\ OQ = OP.$$

Thus, for all values of θ,

$$\sin(180^\circ + \theta) = \sin XOQ = \frac{QM}{OQ} = \frac{-PN}{OP} = -\sin\theta$$

$$\cos(180^\circ + \theta) = \frac{OM}{OQ} = \frac{-ON}{OP} = -\cos\theta$$

$$\tan(180^\circ + \theta) = \frac{QM}{OM} = \frac{-PN}{-ON} = \frac{PN}{ON} = \tan\theta$$

and so,

$$\operatorname{cosec}(180^\circ+\theta) = -\operatorname{cosec}\theta,$$
$$\sec(180^\circ+\theta) = -\sec\theta,$$
$$\cot(180^\circ+\theta) = \cot\theta.$$

26. Ratios of $(270^\circ-\theta)$.

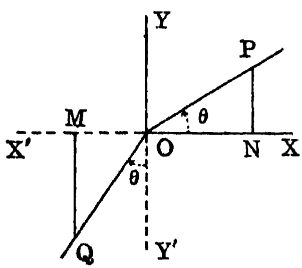


Fig. (i)

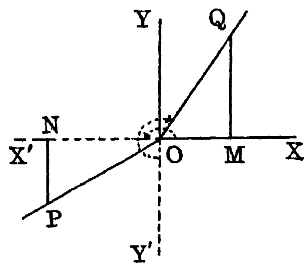


Fig. (ii)

Let $\angle XOP=\theta$ be traced out by a revolving line, and let another revolving line trace out an angle $XOY'=270^\circ$, thereby coming up to the position OY', and then revolve back, tracing out an angle $Y'OQ=\theta$, so that $\angle XOQ=270^\circ-\theta$.

Two figures are given here with OP in the first and third quadrants. The other two may be drawn similarly.

Cut off $OP=OQ$ and drawn PN, QM perpendiculars on XOX'.

Since $\angle XOP=\angle Y'OQ$ in magnitude, we easily derive that $\angle PON=\angle OQM$ in magnitude. Also $OP=OQ$. Hence, the two right-angled triangles OPN and OQM are congruent. Considering signs, we get for all the figures,

$$QM=-ON,\ OM=-PN,\ OQ=OP.$$

Hence, for all values of θ,

$$\sin(270^\circ-\theta)=\sin\angle XOQ=\frac{QM}{OQ}=\frac{-ON}{OP}=-\cos\theta$$

$$\cos(270^\circ-\theta)=\frac{OM}{OQ}=\frac{-PN}{OP}=-\sin\theta$$

$$\tan(270^\circ-\theta)=\frac{QM}{OM}=\frac{-ON}{-PN}=\frac{ON}{PN}=\cot\theta\ ;$$

and thus,

$$\operatorname{cosec}(270^\circ-\theta)=-\sec\theta,$$
$$\sec(270^\circ-\theta)=-\operatorname{cosec}\theta,$$
$$\cot(270^\circ-\theta)=\tan\theta.$$

27. Ratios of $(270^\circ+\theta)$.

We may proceed geometrically as in the previous cases. Otherwise we may proceed as follows :

$$\sin(270^\circ+\theta)=\sin(180^\circ+90^\circ+\theta)=-\sin(90^\circ+\theta)\quad[\text{from § 25}$$
$$=-\cos\theta\quad\cdots\quad\cdots\quad[\text{from § 23}$$

$$\cos(270^\circ+\theta)=\cos(180^\circ+90^\circ+\theta)=-\cos(90^\circ+\theta)$$
$$=-(-\sin\theta)=\sin\theta$$

$$\tan(270^\circ+\theta)=\frac{\sin(270^\circ+\theta)}{\cos(270^\circ+\theta)}=\frac{-\cos\theta}{\sin\theta}=-\cot\theta\ ;$$

and hence,

$$\operatorname{cosec}(270^\circ+\theta)=-\sec\theta,$$
$$\sec(270^\circ+\theta)=\operatorname{cosec}\theta,$$
$$\cot(270^\circ+\theta)=-\tan\theta.$$

Note. The ratios of $180^\circ-\theta$, $180^\circ+\theta$, $270^\circ-\theta$ can be similarly deduced from the formulæ for ratios of $90^\circ\pm\theta$.

28. Ratios of $(360^\circ-\theta)$, $(360^\circ+\theta)$ and $(n.\ 360^\circ\pm\theta)$.

It has already been remarked in Art. 2, Chapter I, that angles which differ by complete multiples of 360°, *i.e.*, by an exact number of complete revolutions, have the final positions of the revolving lines coincident, if the initial lines

are the same. Hence, all the trigonometrical ratios of two such angles must be identical in magnitude as well as in sign.

Thus, trigonometrical ratios of $360° - \theta$ must be same as those of $-\theta$. Hence,

$$\sin(360° - \theta) = \sin(-\theta) = -\sin\theta$$
$$\cos(360° - \theta) = \cos(-\theta) = \cos\theta$$
$$\tan(360° - \theta) = \tan(-\theta) = -\tan\theta, \text{ etc.}$$

Trigonometrical ratios of $360° + \theta$, or of $360° \times n \pm \theta$, where n is an integer, positive of negative, must similarly be same as those of θ, or of $\pm\theta$.

Thus, *in determining trigonometrical ratios of angles, complete multiples of* $360°$ (*i.e.*, 2π) *may be always added or subtracted.*

29. All the above results may, for easy remembrance, be summed up in **a simple rule.**

If θ be associated with an even multiple of 90° by + or − sign, (*e.g.*, $180° - \theta$, $180° + \theta$, $360° - \theta$, $360° + \theta$, etc.) **the ratio is not altered** in form (*i.e.*, sine remains sine, cosine remains cosine, etc.). To determine the sign, assuming θ to be acute, find out the quadrant in which the associated angle lies, and determine the sign according to the rule, "all, sin, tan, cos".

If θ be associated with an odd multiple of 90° by + or − sign, (*e.g.*, $90° - \theta$, $90° + \theta$, $270° - \theta$, $270° + \theta$, etc.) **the ratio is altered** (sine becomes cosine, cosine becomes sine, tangent becomes cotangent, etc.). Moreover, the sign of the result is determined as in the previous paragraph.

Example. *Consider formulæ for tan* $(270° - \theta)$ *and sec* $(180° + \theta)$.

$$270° - \theta = 3.90° - \theta \text{ (multiple of 90° is odd).}$$

Hence, the ratio will be altered, tan changing into cot. Moreover, θ being assumed acute (whether it actually is so or not, it does not matter), $270° - \theta$ falls in the third quadrant, where tan is positive.

Hence, $\tan(270° - \theta) = +\cot\theta$.

$180° + \theta$ has got θ associated with even multiple of $90°$. Hence, the ratio does not alter in form, sec remaining sec. Also, $180° + \theta$ falls in the third quadrant, if θ be assumed acute, where sec (by the rule "all, sin, tan, cos") is negative.

Hence, $\sec(180° + \theta) = -\sec\theta$.

N. B. The angle '$-\theta$' may be written as $0.360° - \theta$, and 0 may be considered *even* in applying the above rule.

Thus, θ being supposed acute, $-\theta$ falls in the fourth quadrant, where cos and sec only are positive. The form of the ratio not changing in this case, $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = +\cos\theta$, etc.

30. Special angles *(outside the first quadrant).*

In Art. 24, putting $\theta = 60°$, $45°$, $30°$ and $0°$ respectively we can deduce the following results :

$$\sin 120° = \sin 60° = \frac{\sqrt{3}}{2}; \qquad \cos 120° = -\cos 60° = -\frac{1}{2}.$$

$$\sin 135° = \sin 45° = \frac{1}{\sqrt{2}}; \qquad \cos 135° = -\cos 45° = -\frac{1}{\sqrt{2}}.$$

$$\sin 150° = \sin 30° = \frac{1}{2}; \qquad \cos 150° = -\cos 30° = -\frac{\sqrt{3}}{2}.$$

$$\sin 180° = \sin 0° = 0; \qquad \cos 180° = -\cos 0° = -1.$$

And similarly from Art. 27 and 28, putting $\theta = 0$,

$$\sin 270° = -\cos 0° = -1; \qquad \cos 270° = \sin 0° = 0;$$

$$\sin 360° = \sin 0° = 0; \qquad \cos 360° = \cos 0° = 1.$$

From the above, we get,

$$\tan 180° = 0; \ \tan 270° = \pm\infty; \ \tan 360° = 0.$$

Examples worked out.

Ex. 1. *Find the value of cot* $(-1575°)$.

$$\cot(-1575°) = -\cot(1575°) = -\cot(4 \times 360° + 135°)$$
$$= -\cot(135°) = -\cot(180° - 45°)$$
$$= \cot 45° = 1.$$

Ex. 2. *Find the value of cot* $\theta - \tan\theta$, where $\theta = \frac{17\pi}{3}$.

$\frac{17\pi}{3} = 6\pi - \frac{\pi}{3}$, and omitting complete multiples of $360°$ *i.e.*, of 2π, whereby trigonometrical ratios are not altered, we get,

$$\cot\frac{17\pi}{3} = \cot\left(-\frac{\pi}{3}\right) = -\cot\frac{\pi}{3} = -\cot 60° = -\frac{1}{\sqrt{3}}.$$

$$\tan\frac{17\pi}{3} = \tan\left(-\frac{\pi}{3}\right) = -\tan\frac{\pi}{3} = -\tan 60° = -\sqrt{3}.$$

$$\therefore\quad \cot\theta - \tan\theta = -\frac{1}{\sqrt{3}} + \sqrt{3} = \frac{3-1}{\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

Ex. 3. *Prove that*

$$\sin(420°)\cos(390°) + \cos(-300°)\sin(-330°) = 1.$$

[H. S. 1962]

$$\text{L. H. side} = \sin(360° + 60°)\cos(360° + 30°)$$
$$+ \cos(-360° + 60°)\sin(-360° + 30°)$$
$$= \sin 60°\cos 30° + \cos 60°\sin 30°$$
$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1.$$

Ex. 4. *Express cot* $(-1358°)$ *in terms of the ratio of a positive angle less than* $45°$.

$$\cot(-1358°) = \cot(-3 \times 360° + 82°)$$
$$= \cot 82° = \cot(90° - 8°)$$
$$= \tan 8°.$$

Note. **Ratios of angles of any magnitude and sign can always be expressed in terms of a ratio of a positive angle less than 45°.**

Ex. 5. *Express*
$$\frac{\cos(90^\circ+\theta)\sec(-\theta)\tan(180^\circ-\theta)}{\sec(360^\circ+\theta)\sin(180^\circ+\theta)\cot(90^\circ-\theta)}$$ *in its simplest form.*

The given expression

$$= \frac{-\sin\theta.\sec\theta.(-\tan\theta)}{\sec\theta.(-\sin\theta).\tan\theta}$$
$$= -1.$$

Examples IV

1. Write down the values of $\sin 150^\circ$, $\cot 840^\circ$, $\operatorname{cosec}(-660^\circ)$ and $\tan(-1125^\circ)$.

2. Find the values of $\sin\left(-\frac{11\pi}{4}\right)$, $\operatorname{cosec}\left(\frac{16\pi}{3}\right)$, $\tan\left(\frac{3\pi}{2}+\frac{\pi}{3}\right)$ and $\cos\left(\frac{5\pi}{2}-\frac{19\pi}{3}\right)$.

3. Evaluate $\sin\left(-1230^\circ\right)-\cos\left\{\left(2n+1\right)\pi+\frac{\pi}{3}\right\}$, where n is a negative integer.

4. Find the value of $\sin\left\{n\pi+(-1)^n\frac{\pi}{3}\right\}$, where n is any integer.

5. Find all the values of

(i) $\tan\left\{\frac{n\pi}{2}+(-1)^n\frac{\pi}{4}\right\}$;

(ii) $\operatorname{cosec}\left\{\frac{n\pi}{2}+(-1)^n\frac{\pi}{6}\right\}$,

where n is any integer.

6. Show that $\cos\left(2m\pi\pm\frac{\pi}{3}\right)$ and $\tan\left(m\pi+\frac{\pi}{6}\right)$ have one value each for all integral values of m.

7. Prove that, n being any integer

(i) $\cos(n\pi + a) = (-1)^n \cos a$.

(ii) $\tan(n\pi - a) = -\tan a$.

8. Prove that

(i) $\cos\theta = -\cos(\theta - 180^\circ)$.

(ii) $\tan\theta = -\cot(\theta - \frac{3}{2}\pi)$.

9. Prove that

(i) $\sin(780^\circ)\cos(390^\circ) - \sin(330^\circ)\cos(-300^\circ) = 1$.

(ii) $\cos 306^\circ + \cos 234^\circ + \cos 162^\circ + \cos 18^\circ = 0$.

(iii) $\dfrac{\sin 250^\circ + \tan 290^\circ}{\cot 200^\circ + \cos 340^\circ} = -1$.

10. Simplify

$$\frac{\sin^3(\pi+\theta)}{\cos^3(\frac{1}{2}\pi+\theta)} \cdot \frac{\tan(2\pi-\theta)}{\operatorname{cosec}^2\theta} \cdot \frac{\sec^2(\pi-\theta)}{\sin(\pi-\theta)}$$

and determine its value when $\theta = 225^\circ$.

11. Prove that

$$\sin(\tfrac{1}{2}\pi+\theta)\cos(\pi-\theta)\cot(\tfrac{3}{2}\pi+\theta)$$
$$= \sin(\tfrac{1}{2}\pi-\theta)\sin(\tfrac{3}{2}\pi-\theta)\cot(\tfrac{1}{2}\pi+\theta).$$

12. Evaluate

(i) $\sin^2\dfrac{\pi}{4} + \sin^2\dfrac{3\pi}{4} + \sin^2\dfrac{5\pi}{4} + \sin^2\dfrac{7\pi}{4}$.

(ii) $\cot\dfrac{\pi}{20}\cot\dfrac{3\pi}{20}\cot\dfrac{5\pi}{20}\cot\dfrac{7\pi}{20}\cot\dfrac{9\pi}{20}$.

(iii) $\sin x + \sin(\pi + x) + \sin(2\pi + x) + \cdots\cdots$ to n terms.

13. If $\tan\theta = \frac{5}{12}$ and $\cos\theta$ is negative, find the value of

$$\frac{\sin\theta + \cos(-\theta)}{\sec(-\theta) + \tan\theta}.$$

14. An angle θ lies between 180° and 270°, and $\operatorname{cosec}\theta = -\frac{5}{3}$. Find $\cot\theta$.

15. Express in terms of ratios of positive angles less than $45°$;

(i) $\cot(-1054°)$. (ii) $\sin(1145°)$.

(iii) $\sec(-1491°)$. (iv) $\cos\frac{35\pi}{9}$.

16. Find the values of θ when,

(i) $\tan\theta = -\sqrt{3}$ and θ lies between $270°$ and $360°$.

(ii) $\cos\theta = -\frac{1}{2}$, and $450° < \theta < 540°$.

17. Solve for θ, giving all the possible values, when $0° < \theta < 360°$;

(i) $\cos\theta + \sqrt{3}\sin\theta = 2$. [*C. U. 1936*]

(ii) $2\sin^2\theta + 3\cos\theta = 0$.

(iii) $3(\sec^2\theta + \tan^2\theta) = 5$.

(iv) $\cot\theta + \tan\theta = 2\sec\theta$.

(v) $1 - 2\sin\theta - 2\cos\theta + \cot\theta = 0$.

18. If A, B, C be angles of a triangle, show that

$$\sin(A+B) - \cos C = \cos(A+B) + \sin C.$$

19. If A, B, C be angles of a triangle, show that

$$\frac{\tan(B+C) + \tan(C+A) + \tan(A+B)}{\tan(\pi - A) + \tan(2\pi - B) + \tan(3\pi - C)} = 1.$$

20. If A, B, C, D be the angles of a quadrilateral, show that

$$\cos\tfrac{1}{2}(A+C) + \cos\tfrac{1}{2}(B+D) = 0.$$

If the quadrilateral be cyclic then

$$\cos A + \cos B + \cos C + \cos D = 0.$$

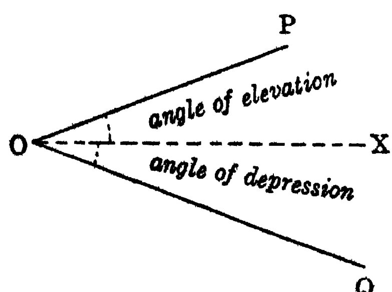
CHAPTER V

SIMPLE PRACTICAL APPLICATIONS OF TRIGONOMETRY

(*Heights and Distances*)

31. One of the most important applications of Trigonometry is in the determination of *heights and distances* of distant objects which are not directly measurable, by observations of angles subtended by those objects at the eye of the observer. These angles may be measured by instruments knows as Sextants or Theodolites or by other angle-measuring instruments. Thus, Trigonometry plays a very important part in *land survey*. It is also extensively used by Astronomers in determining the distances of the heavenly bodies like the sun, moon and stars.

Two angles are very often used in the practical applications of Trigonometry, and they are defined as follows :—

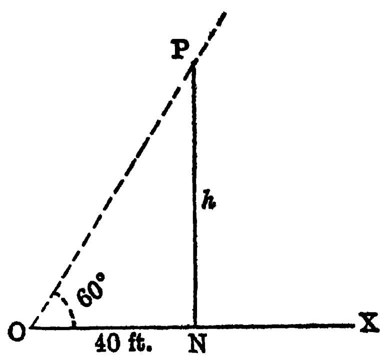


If a horizontal line OX be drawn through O, the eye of an observer, the angle which the line joining O to a point P above OX makes with OX is called the **Angle of Elevation** or *altitude of P* as seen from O.

If Q be below the horizontal line OX, the angle XOQ measured below OX is called the **Angle of Depression** of Q as seen from O.

32. Illustrative Examples.

Ex. 1. *From a distance of 40 feet from the foot of a palm tree in a horizontal field, the angle of elevation of the top of the tree is observed to be 60°. Find the height of the tree.*



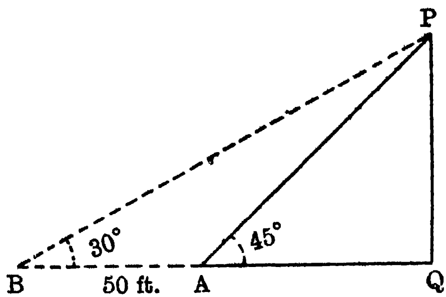
Let h ft. be the height of the tree PN, and $\angle NOP$, the angle of elevation of P as seen from O, where $ON = 40$ ft., is 60°.

Then, $\frac{h}{40} = \tan PON = \tan 60° = \sqrt{3}$;

$\therefore \quad h = 40\sqrt{3}$ ft. $= 69{\cdot}28......$ ft.

Ex. 2. *From one bank of a river, the top of a building just on the opposite bank is observed to have an elevation of 45°. On receding 50 ft. from the bank, perpendicular to its edge, the angle of elevation becomes 30°. Find the breadth of the river and the height of the building.*

AQ being the breadth of the river, PQ the height of the building, $\angle PAQ = 45°$. Also, AB being 50 ft., $\angle PBQ = 30°$.



Now, $\dfrac{BQ}{PQ} = \cot 30°$, $\dfrac{AQ}{PQ} = \cot 45°$.

Hence, subtracting, $\dfrac{AB}{PQ} = \cot 30° - \cot 45°$,

or, $\dfrac{50}{PQ} = \sqrt{3} - 1$;

$\therefore \quad PQ = \dfrac{50}{\sqrt{3}-1} = \dfrac{50(\sqrt{3}+1)}{2} = 68{\cdot}3$ ft. nearly.

Also, $\dfrac{AQ}{PQ} = \cot 45° = 1$; $\therefore\ AQ = PQ = 68{\cdot}3$ ft.

Thus, the breadth of the river and the height of the building are both 68·3 ft. nearly.

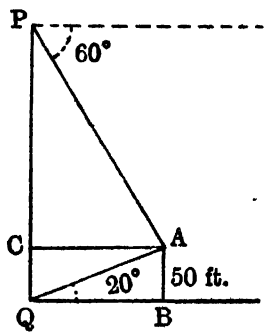
Ex. 3. *The angles of depression and elevation of the top of a tower 50 ft. high from the top and bottom of a second tower are 60° and 20° respectively. Find the height of the second tower to the nearest foot.* [Given cot 20° = 2·747.]

PQ is the second tower, and $\angle XPA = 60°$, $\angle BQA = 20°$, $AB = 50$ ft., AC is parallel to BQ or PX, so that $\angle PAC$ = the alternate angle $XPA = 60°$.

Now, $\dfrac{QB}{AB} = \cot 20^\circ$; $\therefore$ $QB = AB \cot 20^\circ$.

Also, $\dfrac{PC}{CA} = \tan PAC = \tan 60^\circ$;

$$\therefore \quad PC = CA \tan 60^\circ = QB \tan 60^\circ$$
$$= AB \cot 20^\circ \tan 60^\circ.$$



$$\therefore \text{ height } PQ = PC + CQ = PC + AB$$
$$= AB (\cot 20^\circ \tan 60^\circ + 1)$$
$$= 50 (2{\cdot}747 \times \sqrt{3} + 1)$$
$$= 287{\cdot}8 \ldots \text{ ft.} = 288 \text{ ft. nearly.}$$

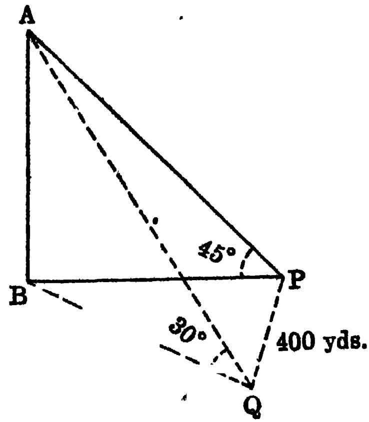
Ex. 4. *The elevation of a hill from a place P due East of it is 45°, and at a place Q due South of P, the elevation is 30°. If the distance PQ be 400 yds., find the height of the hill.*

A is the top of the hill, B is the point vertically below it on the ground. BP is due East, PQ is due South, so that BPQ is a right angle. Also ABP and ABQ are both right angles.

Now, $\dfrac{BQ}{AB} = \cot AQB = \cot 30^\circ = \sqrt{3}$,

and, $\dfrac{BP}{AB} = \cot APB = \cot 45^\circ = 1$,

Hence, $BQ = AB\sqrt{3}$, $BP = AB$,



and $PQ^2 = BQ^2 - BP^2 = AB^2\ (3-1) = 2AB^2$.

$$\therefore \quad AB = \frac{PQ}{\sqrt{2}} = \frac{PQ}{2} \cdot \sqrt{2} = 200\sqrt{2} = 283 \text{ yds. nearly.}$$

Examples V

1. From the top of a tower by the seaside, 100 feet high, it was observed that the angle of depression of the bottom of a ship at anchor was 30°. Find the distance of the ship from the bottom of the tower.

2. Two straight roads, which cross one another, meet a river with straight course at angles 60° and 30° respectively. If it be 3 miles by the longer of the two roads, from the crossing to the river, how far is it by the shorter? If there be a foot-path which goes the shortest way from the crossing to the river, what is the distance by it?

3. Two poles are of equal height; a person standing midway between the line joining their bases observes the

elevation of the poles to be 30°. After walking 40 feet towards one of them, he observes that the same pole now subtends an angle of 60°. Find their height and the distance between them.

4. A straight palm tree 60 feet high, is broken by the wind but not completely separated, and its upper part meets the ground at an angle of 30°. Find the distance of the point where the top of the tree meets the ground, from the root, and also the height at which the tree is broken.

5. Two posts are 120 ft. apart, and the height of one is double that of the other. From the middle point of the line joining their feet, an observer finds the angular elevations of their tops to be complementary. Find the height of the shorter post. [*H. S. 1961 Com.*]

6. The Bally bridge subtends an angle of 45° at a given point at the edge of the river ; 800 yds. higher up, it subtends an angle of 30°. The course of the river here is straight and perpendicular to the bridge. Find the length of the bridge.

7. The height of a house subtends a right angle at an opposite window, the top being 60° above a horizontal straight line through the window ; find the height of the house, taking the breadth of the street to be 30 feet.

8. From an aeroplane vertically over a straight road, the angles of depression of two consecutive milestones are observed to be 45° and 60° ; find the height of the aeroplane.

9. From a ship sailing due South-East at the rate of 5 miles an hour, a light-house is observed to be 30° North of East, and after 4 hours, it is seen due North ; find the distance of the light-house from the final position of the ship.

10. The shadow of a tower standing on a level plane is found to be 40 feet longer when the sun's altitude is 45° than when it is 60°. Find the height of the tower.

11. From the lower window of a house the angular elevation of a church-steeple is found to be 45° and from a window 20 feet above, the elevation is 30°. How far is the church from the house?

12. A light-house facing East sends out a fan-shaped beam of light extending from S. E. to N. E. An observer sailing due North, after meeting the light continues to see it for $10\sqrt{2}$ minutes. When leaving the fan of light, the ship is 10 miles from the light-house. Find the speed of the ship.

13. A pole 100 ft. high stands vertically at the centre of a horizontal equilateral triangle, each side of which subtends an angle of 60° at the top of the pole. Find the side of the triangle.

14. Two chimneys are of equal height. A person standing between them in the line joining their bases observes the elevation of the nearer one to be 60°. After walking 80 feet in a direction at right angles to the line joining their bases, he observes the elevations of the two to be 45° and 30° respectively. Find the height and the distance between them.

15. At the foot of a mountain the elevation of its summit is 45°; after ascending 1 mile towards the mountain up an incline of 30°, the elevation changes to 60°. Find the height of the mountain.

16. From a station, two light-houses A and B are seen in directions North and 30° East of North respectively; if A were one-third as far off as it really is, it would appear due West of B. If the distance of B from the station be 10 miles, find the distance of B from A.

17. A person walking along a straight road observes a tall tree standing in front of a tower, both being on the road before him. The elevation of the top of the tower is 45°, and of the top of the tree 30°; on advancing 100 feet, he finds the tower and the tree to have the same elevation 60°; supposing the height of the eye of the man to be 5 feet, find the height of the tower and of the tree.

18. A man on the top of a rock rising on a seashore, observes a boat coming towards it at an angle of depression 30° ; 10 minutes later the angle of depression is 60°. The height of the rock being 4000 feet, find the speed of the boat in miles per hour.

19. A person walking along a straight level road observes the elevation of the top of a hill to be 60° when he is nearest the hill, and after walking 200 yards in a direction perpendicular to the direction of the hill from this point, observes the elevation to be 30°. Find the approximate height of the hill.

20. A square tower stands on a horizontal plane. From a point in this plane, only three of its upper corners are visible, and their angles of elevation are 45°, 60°, 45°. Find the ratio of the height of the tower to its breadth.

21. Two wheels, the sum of whose radii is 10 feet, are placed flatly on a table with their centres at a distance of 20 ft. An endless string, quite stretched, is partly wrapped round the wheels and crosses itself between them. Show that the length of the string is nearly 76·5 feet.

22. On a still day, from a station A an airship is observed due north at an elevation of 60°, while from a station B it is observed due east at an elevation of 45°. At this instant of observation, a parachute message is dropped from the airship, and the observer at A has to walk a mile to reach the message. Find the distance between the two stations.

23. From the foot of a column the angle of elevation of the top of a tower is 45° and from the top of the column the angle of depression of the bottom of the tower is 30°. A man walks 10 ft. from the bottom of the column towards the tower and notices the angle of elevation of its top to be 60°. Find the height of the column.

CHAPTER VI

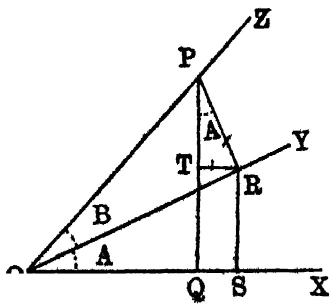
COMPOUND ANGLES

33. *To prove that*

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$

when A and B are *positive* and *acute* and $(A+B) \leq 90°$.



Let a revolving line starting from the position OX trace out an angle $XOY = A$ and then revolving further, trace out an angle $YOZ = B$; then $\angle XOZ = A + B$.

In OZ, the *bounding line of the compound angle* $A+B$, take any point P and draw PQ and PR perpendicular to OX and OY respectively; also draw RS and RT perpendicular to OX and PQ respectively.

From the right-angled $\triangle POQ$,

$$\sin(A+B) = \frac{PQ}{OP} = \frac{QT+TP}{OP} = \frac{RS+PT}{OP} = \frac{RS}{OP} + \frac{PT}{OP}$$

$$= \frac{RS}{OR} \cdot \frac{OR}{OP} + \frac{PT}{PR} \cdot \frac{PR}{OP}$$

$$= \sin A \cos B + \cos TPR \cdot \sin B.$$

Now, $\angle TPR = 90° - \angle TRP = \angle TRO = \angle ROS = A$.

$\therefore \sin(A+B) = \sin A \cos B + \cos A \sin B.$

Again,

$$\cos(A+B) = \frac{OQ}{OP} = \frac{OS-QS}{OP} = \frac{OS-TR}{OP} = \frac{OS}{OP} - \frac{TR}{OP}$$

$$= \frac{OS}{OR}\cdot\frac{OR}{OP} - \frac{TR}{PR}\cdot\frac{PR}{OP}$$

$$= \cos A \cos B - \sin TPR \,.\, \sin B$$

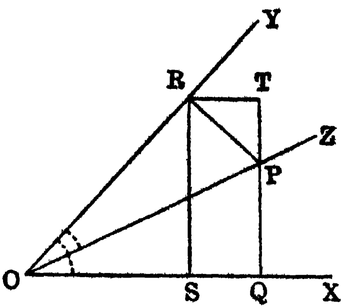
$$= \cos A \cos B - \sin A \sin B.$$

34. *To prove that*

$$\mathbf{\sin(A-B) = \sin A \cos B - \cos A \sin B}$$

$$\mathbf{\underline{\cos(A-B) = \cos A \cos B + \sin A \sin B,}}$$

when A and B are *positive* and *acute*, and $A > B$.



Let a revolving line start from the position OX and trace out an angle $XOY = A$ and then revolving back trace out an angle $YOZ = B$; then $\angle XOZ = A - B$.

In OZ, the *bounding line of the compound angle* $A - B$, take any point P, and draw PQ and PR perpendicular to OX and OY respectively; and draw RS and RT perpendicular to OX and QP produced respectively.

From the right-angled ΔPOQ,

$$\sin(A-B) = \frac{PQ}{OP} = \frac{TQ-PT}{OP} = \frac{RS-PT}{OP} = \frac{RS}{OP} - \frac{PT}{OP}$$

$$=\frac{RS}{OR}\cdot\frac{OR}{OP}-\frac{PT}{PR}\cdot\frac{PR}{OP}$$

$$=\sin A\cos B-\cos TPR\,.\,\sin B.$$

But $\angle TPR=90^\circ-\angle TRP=\angle YRT=\angle YOX=A.$

$\therefore\ \sin(A-B)=\sin A\cos B-\cos A\sin B.$

Again,

$$\cos(A-B)=\frac{OQ}{OP}=\frac{OS+SQ}{OP}=\frac{OS+RT}{OP}=\frac{OS}{OP}+\frac{RT}{OP}$$

$$=\frac{OS}{OR}\cdot\frac{OR}{OP}+\frac{RT}{RP}\cdot\frac{RP}{OP}$$

$$=\cos A\cos B+\sin TPR\,.\,\sin B$$

$$=\cos A\cos B+\sin A\sin B.$$

Obs. In the above Geometrical proofs, it is assumed that the angles A, B, $A+B$ are all less than a right angle and that $A-B$ is positive. If the angles are not so restricted, the same method of proof (there being some modifications in the figures) will apply, due attention being paid to the signs of the quantities involved.*

Thus, the above formulæ are perfectly general.

Note 1. The sum or difference of two or more angles is called a *Compound angle*; such as $A+B$, $A-B$, $A+B+C$ etc.

The expansions $\sin(A\pm B)$ and $\cos(A\pm B)$ are generally called the "*Addition formulæ* or *Addition and Subtraction Theorems*".

Note. Assuming the truth of the above formulæ for acute angles, they can be shown to be true for angles of any magnitude, as follows

Let us consider $\sin(A+B)$.

Let A and B be acute and $A+B<90^\circ$.

Let $A_1=90^\circ+A$; $B_1=B$.

Now, $\sin(A_1+B_1)=\sin\{(90^\circ+A)+B\}=\sin\{90^\circ+(A+B)\}$

$=\cos(A+B)=\cos A\cos B-\sin A\sin B$ [*by Art. 33*]

$=\sin(90^\circ+A)\cos B+\cos(90^\circ+A)\sin B$

$=\sin A_1\cos B_1+\cos A_1\sin B_1.$

*See *Appendix*, *Arts. 2-4.*

Again, let $A_2=-A$, $B_2=B$.

Then, $\sin(A_2+B_2)=\sin(-A+B)=-\sin(A-B)$

$$=-\sin A\cos B+\cos A\sin B, \qquad \text{[by Art. 34]}$$
$$=\sin(-A)\cos B+\cos(-A)\sin B$$
$$=\sin A_2\cos B_2+\cos A_2\sin B_2.$$

Thus, the above formulæ remain true if any of the two angles is either increased by 90°, or has its sign changed.

In the same way it may be shown that the other three formulæ for $\cos(A+B)$, $\sin(A-B)$ and $\cos(A-B)$ will continue to hold good unchanged in form, if any of the two angles be either increased by 90° or has its sign changed.

Now starting from positive acute-angled values of A and B, combining the two processes of increasing one of the angles by 90°, and reversing the sign of any one, we can arrive at values of A and B of any magnitude, positive, or negative, and the four formulæ will still hold good.

Thus, the formulae for $\sin(A\pm B)$ and $\cos(A\pm B)$ are perfectly general.

35. Ex. 1. *Find the values of*

$\sin 75°$, $\cos 75°$, $\sin 15°$ and $\cos 15°$.

$$\sin 75°=\sin(45°+30°)=\sin 45°\cos 30°+\cos 45°\sin 30°$$
$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$$\cos 75°=\cos(45°+30°)=\cos 45°\cos 30°-\sin 45°\sin 30°$$
$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

$\sin 15°=\sin(45°-30°)=\sin 45°\cos 30°-\cos 45°\sin 30°$
and $\cos 15°=\cos(45°-30°)=\cos 45°\cos 30°+\sin 45°\sin 30°$;
therefore, substituting the values of $\sin 45°$, $\cos 45°$ etc. as before, we get

$$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}} \quad \text{and} \quad \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Note. The values of sin 15° and cos 15° can also be deduced from the fact that

$$\sin 15^\circ = \sin(90^\circ - 75^\circ) = \cos 75^\circ$$

and $$\cos 15^\circ = \cos(90^\circ - 75^\circ) = \sin 75^\circ.$$

Ex. 2. *Show that*

(i) $$\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A.$$

(ii) $$\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$$

(i) Left side

$$\begin{aligned}
&= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B) \\
&= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B \\
&= \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B \\
&= \sin^2 A - \sin^2 B \\
&= (1 - \cos^2 A) - (1 - \cos^2 B) = \cos^2 B - \cos^2 A.
\end{aligned}$$

(ii) Left side

$$\begin{aligned}
&= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B) \\
&= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B \\
&= \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B \\
&= \cos^2 A - \sin^2 B \\
&= (1 - \sin^2 A) - (1 - \cos^2 B) = \cos^2 B - \sin^2 A.
\end{aligned}$$

Note. The results of Ex. 1 and Ex. 2 are very useful and should be carefully remembered.

36. *To prove that*

(i) $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(ii) $$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

We have

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}.$$

Now, dividing the numerator and denominator by $\cos A\cos B$, we have

$$\tan(A+B)=\frac{\frac{\sin A\cos B}{\cos A\cos B}+\frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B}-\frac{\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

Again,

$$\tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B}.$$

Now, dividing the numerator and denominator by $\cos A\cos B$, we have, as before,

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

37. *To prove that*

$$\textbf{(i)}\ \cot(A+B)=\frac{\cot A\cot B-1}{\cot B+\cot A}$$

$$\textbf{(ii)}\ \cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A}.$$

$$\cot(A+B)=\frac{\cos(A+B)}{\sin(A+B)}=\frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}.$$

Now, dividing the numerator and denominator by $\sin A\sin B$, we have,

$$\cot(A+B)=\frac{\frac{\cos A\cos B}{\sin A\sin B}-\frac{\sin A\sin B}{\sin A\sin B}}{\frac{\sin A\cos B}{\sin A\sin B}+\frac{\cos A\sin B}{\sin A\sin B}}$$

$$=\frac{\cot A\cot B-1}{\cot B+\cot A}.$$

$$\cot(A-B)=\frac{\cos(A-B)}{\sin(A-B)}=\frac{\cos A\cos B+\sin A\sin B}{\sin A\cos B-\cos A\sin B}.$$

Now, dividing the numerator and denominator by $\sin A\sin B$, we have, as before,

$$\cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A}.$$

38. Ex. 1. *Find the values of tan 75° and tan 15°.*

$$\textbf{tan 75}^\circ=\tan(45^\circ+30^\circ)=\frac{\tan 45^\circ+\tan 30^\circ}{1-\tan 45^\circ\tan 30^\circ}$$

$$=\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}=\frac{(\sqrt{3}+1)(\sqrt{3}+1)}{3-1}$$

$$=\frac{4+2\sqrt{3}}{2}=\mathbf{2+\sqrt{3}}.$$

$$\textbf{tan 15}^\circ=\tan(45^\circ-30^\circ)=\frac{\tan 45^\circ-\tan 30^\circ}{1+\tan 45^\circ\tan 30^\circ}$$

$$=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$=\frac{4-2\sqrt{3}}{2}=\mathbf{2-\sqrt{3}}.$$

Ex. 2. *Show that*

(i) $\tan(45^\circ+A)=\dfrac{1+\tan A}{1-\tan A}.$

(ii) $\tan(45^\circ-A)=\dfrac{1-\tan A}{1+\tan A}.$

(i) Left side $=\dfrac{\tan 45^\circ+\tan A}{1-\tan 45^\circ\tan A}=\dfrac{1+\tan A}{1-\tan A}.$

(ii) This result follows similarly.

Ex. 3. *Show that*

$$\cot 2A + \tan A = \operatorname{cosec} 2A. \qquad [\text{ C. U. 1947 }]$$

Left side $= \dfrac{\cos 2A}{\sin 2A} + \dfrac{\sin A}{\cos A} = \dfrac{\cos 2A \cos A + \sin 2A \sin A}{\sin 2A \cos A}$

$$= \frac{\cos(2A - A)}{\sin 2A \cos A} = \frac{\cos A}{\sin 2A \cos A} = \frac{1}{\sin 2A}$$

39. *To find the expansions of*

(i) sin (A+B+C)

(ii) cos (A+B+C)

(iii) tan (A+B+C)

(i) $\sin(A+B+C)$

$$= \sin\{(A+B)+C\}$$
$$= \sin(A+B)\cos C + \cos(A+B)\sin C$$
$$= (\sin A \cos B + \cos A \sin B)\cos C + (\cos A \cos B - \sin A \sin B)\sin C$$
$$= \sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B - \sin A \sin B \sin C.$$

Note 1. The expansion of $\sin(A+B+C)$ can be easily put in the form

$$\cos A \cos B \cos C\,(\tan A + \tan B + \tan C - \tan A \tan B \tan C).$$

(ii) $\cos(A+B+C)$

$$= \cos\{(A+B)+C\}$$
$$= \cos(A+B)\cos C - \sin(A+B)\sin C$$
$$= (\cos A \cos B - \sin A \sin B)\cos C - (\sin A \cos B + \cos A \sin B)\sin C$$
$$= \cos A \cos B \cos C - \cos A \sin B \sin C - \cos B \sin C \sin A - \cos C \sin A \sin B.$$

Note 2. The expansion of $\cos(A+B+C)$ can be easily put in the form

$$\cos A \cos B \cos C\,(1 - \tan B \tan C - \tan C \tan A - \tan A \tan B).$$

(iii) **tan (A+B+C)**

$$= \tan \{(A+B)+C\}$$

$$= \frac{\tan (A+B) + \tan C}{1 - \tan (A+B) \tan C}$$

$$= \frac{\dfrac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \dfrac{\tan A + \tan B}{1 - \tan A \tan B} \tan C}$$

$$= \frac{\mathbf{\tan A + \tan B + \tan C - \tan A \tan B \tan C}}{\mathbf{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}}.$$

Note 3. The expansion of $\tan (A+B+C)$ can also be obtained thus,

$$\tan (A+B+C) = \frac{\sin (A+B+C)}{\cos (A+B+C)}.$$

Now, write down the expansions of $\sin (A+B+C)$ and $\cos (A+B+C)$ and divide the numerator and denominator by $\cos A \cos B \cos C$ or simply write down the expansions of $\sin (A+B+C)$ and $\cos (A+B+C)$ as given in Notes 1 and 2.

Obs. Formulæ for the Trigonometrical functions of the sum of four, five or more angles can be similarly obtained.

Examples VI

Show that (*Ex. 1 to 20*) :—

1. (i) $\sin (A-B) = \frac{16}{65}$ and $\cos (A+B) = \frac{33}{65}$,

if A and B are acute and if $\sin A = \frac{3}{5}$, $\cos B = \frac{12}{13}$.

(ii) $\cos 68° 20' \cos 8° 20' + \cos 81° 40' \cos 21° 40' = \frac{1}{2}$.

(iii) $\sec (x-y) = \frac{85}{84}$, if $\sec x = \frac{17}{8}$, $\operatorname{cosec} y = \frac{5}{4}$.

2. (i) $\sin A \sin (B-C) + \sin B \sin (C-A)$
$+ \sin C \sin (A-B) = 0.$

(ii) $\cos A \sin (B-C) + \cos B \sin (C-A)$
$+ \cos C \sin (A-B) = 0.$

(iii) $\sin (B+C) \sin (B-C) + \sin (C+A) \sin (C-A)$
$+ \sin (A+B) \sin (A-B) = 0.$

(iv) $\sin(\alpha-\theta)\sin(\beta-\gamma)+\sin(\beta-\theta)\sin(\gamma-\alpha)$
$+\sin(\gamma-\theta)\sin(\alpha-\beta)=0.$

3. $\cos(60^\circ-A)\cos(30^\circ-B)-\sin(60^\circ-A)\sin(30^\circ-B)$
$=\sin(A+B).$

4. (i) $\sin(n+1)x\cos(n-1)x-\cos(n+1)x\sin(n-1)x$
$=\sin 2x.$

(ii) $\sin 2\theta\cos\theta+\cos 2\theta\sin\theta$
$=\sin 4\theta\cos\theta-\cos 4\theta\sin\theta.$

5. $\dfrac{\sin B}{\sin A}=\dfrac{\sin(2A+B)}{\sin A}-2\cos(A+B).$

6. $\dfrac{\sin(B-C)}{\cos B\cos C}+\dfrac{\sin(C-A)}{\cos C\cos A}+\dfrac{\sin(A-B)}{\cos A\cos B}=0.$

7. $\dfrac{\sin(B-C)}{\sin B\sin C}+\dfrac{\sin(C-A)}{\sin C\sin A}+\dfrac{\sin(A-B)}{\sin A\sin B}=0.$

8. $\tan(A+B)\tan(A-B)=\dfrac{\sin^2 A-\sin^2 B}{\cos^2 A-\sin^2 B}.$

9. $\tan^2 A-\tan^2 B=\dfrac{\sin(A+B)\sin(A-B)}{\cos^2 A\cos^2 B}.$

10. (i) $\dfrac{\tan(\alpha+\beta)-\tan\alpha}{1+\tan(\alpha+\beta)\tan\alpha}=\tan\beta.$

(ii) If $A+B+C=\pi$ and $\cos A=\cos B\cos C$, show that $\tan A=\tan B+\tan C.$ [*C. U. 1942*]

11. $1+\tan 2\theta\tan\theta=\sec 2\theta.$

12. $\cot\theta-\cot 2\theta=\operatorname{cosec} 2\theta.$

$\tan 20^\circ+\tan 25^\circ+\tan 25^\circ\tan 20^\circ=1.$

14. (i) $\tan(45^\circ+A)=\dfrac{\cos A+\sin A}{\cos A-\sin A}.$

(ii) $\sqrt{2}\sin(45^\circ+A)=\sin A+\cos A.$

15. $\dfrac{\cos 8^\circ+\sin 8^\circ}{\cos 8^\circ-\sin 8^\circ}=\tan 53^\circ.$

16. $\tan(45^\circ + A)\tan(45^\circ - A) = 1.$

17. $\tan(A+B) + \tan(A-B) = \dfrac{\sin 2A}{\cos^2 A - \sin^2 B}.$

18. $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}.$

19. $\cot(45^\circ + x) = \dfrac{\cot x - 1}{\cot x + 1} = \dfrac{\cos x - \sin x}{\cos x + \sin x}.$

20. $\sec(x+y) = \dfrac{\sec x \sec y}{1 - \tan x \tan y}.$

21. Find the expansions of

$\sin(A - B + C)$ and $\tan(A - B - C)$.

22. Express $\cot(A + B + C)$ in terms of

$\cot A$, $\cot B$, $\cot C$.

23. (i) If $a\cos(x + \alpha) = b\cos(x - \alpha)$, prove that

$(a + b)\tan x = (a - b)\cot \alpha.$

ii) If $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$, show that

$1 + \cot\alpha\tan\beta = 0.$ [*C. U. 1939*]

(iii) If $A + B + C = \pi$ and $\cos A = \cos B \cos C$,

then $\cot B \cot C = \frac{1}{2}$.

24. If $\tan\theta = \dfrac{a\sin x + b\sin y}{a\cos x + b\cos y}$,

then $a\sin(\theta - x) + b\sin(\theta - y) = 0.$

25. An angle θ is divided into two parts α, β such that $\tan\alpha : \tan\beta = x : y$; prove that

$$\sin(\alpha - \beta) = \frac{x - y}{x + y}\sin\theta.$$

26. If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$, show that $\Sigma\cos\alpha = 0$ and $\Sigma\sin\alpha = 0$.

CHAPTER VII

TRANSFORMATION OF PRODUCTS AND SUMS

40. Transformation of products into sums or differences.

We have from Arts. 33 and 34,

$$\sin A \cos B + \cos A \sin B = \sin (A+B) \quad \cdots \quad (1)$$

$$\sin A \cos B - \cos A \sin B = \sin (A-B) \quad \cdots \quad (2)$$

Adding (1) and (2), we get

$$2 \sin A \cos B = \sin (A+B) + \sin (A-B). \quad \cdots \quad (3)$$

Subtracting (2) from (1), we get

$$2 \cos A \sin B = \sin (A+B) - \sin (A-B). \quad \cdots \quad (4)$$

Again, from Arts. 33 and 34, we have,

$$\cos A \cos B - \sin A \sin B = \cos (A+B) \quad \cdots \quad (5)$$

$$\cos A \cos B + \sin A \sin B = \cos (A-B). \quad \cdots \quad (6)$$

Adding (5) and (6), we get

$$2 \cos A \cos B = \cos (A+B) + \cos (A-B). \quad \cdots \quad (7)$$

Subtracting (5) from (6), we get

$$2 \sin A \sin B = \cos (A-B) - \cos (A+B). \quad \ldots \quad (8)$$

Thus, we have the following formulæ for transforming a *product* of two sines and cosines into the *sum* or the *difference* of two sines or two cosines.

$$\mathbf{2 \sin A \cos B = \sin (A+B) + \sin (A-B).} \quad \cdots \quad \text{(I)}$$

$$\mathbf{2 \cos A \sin B = \sin (A+B) - \sin (A-B).} \quad \cdots \quad \text{(II)}$$

$$\mathbf{2 \cos A \cos B = \cos (A+B) + \cos (A-B).} \quad \cdots \quad \text{(III)}$$

$$\mathbf{2 \sin A \sin B = \cos (A-B) - \cos (A+B).} \quad \cdots \quad \text{(IV)}$$

41. Transformation of sums or differences into products.

Let $A+B=C$, and $A-B=D$,

then $$A=\frac{C+D}{2} \text{ and } B=\frac{C-D}{2}.$$

Making these substitutions for A and B in the results (3), (4), (7), (8) of Art. 40 and noting that the relation (8) can be written as

$$\cos(A+B)-\cos(A-B)=-2\sin A\sin B$$
$$=2\sin A\sin(-B),$$

we have the following four formulæ for transforming the *sum* or the *difference* of **two sines** only or **two cosines** only into a product of sines and cosines.

$$\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}. \quad \cdots \quad (I)$$

$$\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}. \quad \cdots \quad (II)$$

$$\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}. \quad \cdots \quad (III)$$

$$\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}. \quad \cdots \quad (IV)$$

Note The following concise verbal statement of the above four formulæ is sometimes very convenient

(i) *sine* + *sine* = 2 *sin* ($\frac{1}{2}$ sum). *cos* ($\frac{1}{2}$ diff.)

(ii) *sine* − *sine* = 2 *cos* ($\frac{1}{2}$ sum). *sin* ($\frac{1}{2}$ diff.).

(iii) *cos* + *cos* = 2 *cos* ($\frac{1}{2}$ sum). *cos* ($\frac{1}{2}$ diff.).

(iv) *cos* − *cos* = 2 *sin* ($\frac{1}{2}$ sum). *sin* ($\frac{1}{2}$ diff. reversed).

42. Ex. 1. *Prove that*

(i) $\cos 20^\circ \cos 40^\circ \cos 80^\circ=\frac{1}{8}$.

(ii) $\cos 80^\circ+\cos 40^\circ-\cos 20^\circ=0$.

(*i*) Left side $= \frac{1}{2}.\cos 20^\circ\,(2\cos 40^\circ \cos 80^\circ)$

$$= \tfrac{1}{2}\cos 20^\circ\,(\cos 120^\circ + \cos 40^\circ)$$
$$= \tfrac{1}{2}\cos 20^\circ\,(-\tfrac{1}{2} + \cos 40^\circ)$$
$$= -\tfrac{1}{4}\cos 20^\circ + \tfrac{1}{2}\cos 20^\circ \cos 40^\circ$$
$$= -\tfrac{1}{4}\cos 20^\circ + \tfrac{1}{4}(\cos 60^\circ + \cos 20^\circ)$$
$$-\tfrac{1}{4}\cos 20^\circ + \tfrac{1}{4}(\tfrac{1}{2} + \cos 20^\circ)$$
$$= \tfrac{1}{8}.$$

(*ii*) Left side $= (\cos 80^\circ + \cos 40^\circ) - \cos 20^\circ$

$$= 2\cos 60^\circ \cos 20^\circ - \cos 20^\circ$$
$$= 2.\tfrac{1}{2}\cos 20^\circ - \cos 20^\circ = 0.$$

Ex. 2. *Show that*

$$\frac{\sin\theta + \sin 2\theta + \sin 4\theta + \sin 5\theta}{\cos\theta + \cos 2\theta + \cos 4\theta + \cos 5\theta} = \tan 3\theta.$$

Numerator $= (\sin 5\theta + \sin\theta) + (\sin 4\theta + \sin 2\theta)$

$$= 2\sin 3\theta\cos 2\theta + 2\sin 3\theta\cos\theta$$
$$= 2\sin 3\theta\,(\cos 2\theta + \cos\theta),$$

Denominator $= (\cos 5\theta + \cos\theta) + (\cos 4\theta + \cos 2\theta)$

$$= 2\cos 3\theta\cos 2\theta + 2\cos 3\theta\cos\theta$$
$$= 2\cos 3\theta\,(\cos 2\theta + \cos\theta).$$

$\therefore$ Left side $= \dfrac{2\sin 3\theta\,(\cos 2\theta + \cos\theta)}{2\cos 3\theta\,(\cos 2\theta + \cos\theta)} = \dfrac{\sin 3\theta}{\cos 3\theta} = \tan 3\theta.$

Ex. 3. *Express* $4\cos A\cos B\cos C$ *as the sum of four cosines.*

$4\cos A\cos B\cos C = 2\cos A\,(2\cos B\cos C)$

$$= 2\cos A\,\{\cos(B+C) + \cos(B-C)\}$$
$$= 2\cos A\cos(B+C) + 2\cos A.\cos(B-C)$$
$$= \cos(A+B+C) + \cos(A-B-C)$$
$$+ \cos(A+B-C) + \cos(A-B+C)$$

Ex. 4. *Express as the product of three sines*

$$\sin(B+C-A) + \sin(C+A-B) + \sin(A+B-C) - \sin(A+B+C).$$

Grouping together the first two terms and grouping together the last two terms, the given expression

$$=2\sin C\cos(B-A)+2\cos(A+B)\sin(-C)$$
$$=2\sin C\{\cos(B-A)-\cos(A+B)\}$$
$$=2\sin C(2\sin B\sin A)$$
$$=4\sin A\sin B\sin C.$$

Examples VII

Prove that (*Ex. 1 to 17*) :—

1. $\dfrac{\sin A+\sin B}{\sin A-\sin B}=\tan\dfrac{A+B}{2}\cot\dfrac{A-B}{2}.$

2. $\dfrac{\cos A+\cos B}{\cos B-\cos A}=\cot\dfrac{A+B}{2}\cot\dfrac{A-B}{2}.$

3. $\cos 20^\circ+\cos 100^\circ+\cos 140^\circ=0.$

4. $\sin\theta\sin(60^\circ-\theta)\sin(60^\circ+\theta)=\frac{1}{4}\sin 3\theta.$

5. $\cos\theta\cos(60^\circ-\theta)\cos(60^\circ+\theta)=\frac{1}{4}\cos 3\theta.$

6. $(\sin 3a+\sin a)\sin a+(\cos 3a-\cos a)\cos a=0.$

7. $\cos(A-D)\sin(B-C)+\cos(B-D)\sin(C-A)$
$\qquad\qquad+\cos(C-D)\sin(A-B)=0.$

8. $\cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ=\frac{1}{16}.$

9. $\sin 20^\circ\sin 40^\circ\sin 60^\circ\sin 80^\circ=\frac{3}{16}.$

10. $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\tan\dfrac{A+B}{2}.$

11. $\dfrac{\sin A-\sin B}{\cos B-\cos A}=\cot\dfrac{A+B}{2}.$

12. $\dfrac{\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta}=\tan 4\theta.$

13. $\dfrac{\sin 2A+\sin 5A-\sin A}{\cos 2A+\cos 5A+\cos A}=\tan 2A.$

14. $$\frac{\sin(\alpha+\beta)-2\sin\alpha+\sin(\alpha-\beta)}{\cos(\alpha+\beta)-2\cos\alpha+\cos(\alpha-\beta)}=\tan\alpha.$$

15. $$\frac{\cos 7\alpha+\cos 3\alpha-\cos 5\alpha-\cos\alpha}{\sin 7\alpha-\sin 3\alpha-\sin 5\alpha+\sin\alpha}=\cot 2\alpha.$$

16. $$\sin 2A+\sin 2B+\sin 2C-\sin 2(A+B+C)$$
$$=4\sin(B+C)\sin(C+A)\sin(A+B).$$

17. $$\cos A+\cos B+\cos C+\cos(A+B+C)$$
$$=4\cos\frac{B+C}{2}\cos\frac{C+A}{2}\cos\frac{A+B}{2}.$$

18. If $\sin x=k\sin y$, prove that
$$\tan\tfrac{1}{2}(x-y)=\frac{k-1}{k+1}\tan\tfrac{1}{2}(x+y).$$

19. If $\cos x+\cos y=\frac{1}{3}$ and $\sin x+\sin y=\frac{1}{4}$, prove that $\tan\frac{1}{2}(x+y)=\frac{3}{4}$.

20. If $x\cos\alpha+y\sin\alpha=k=x\cos\beta+y\sin\beta$, prove that
$$\frac{x}{\cos\frac{1}{2}(\alpha+\beta)}=\frac{y}{\sin\frac{1}{2}(\alpha+\beta)}=\frac{k}{\cos\frac{1}{2}(\alpha-\beta)}$$

21. If $\sin\theta+\sin\phi=a$, $\cos\theta+\cos\phi=b$, prove that
$$\tan\frac{\theta-\phi}{2}=\pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}.$$

22. Prove that $\dfrac{\cos 10^\circ-\sin 10^\circ}{\cos 10^\circ+\sin 10^\circ}=\tan 35^\circ.$

[*Note that* $\sin\theta=\cos(90^\circ-\theta)$ *and* $\cos\theta=\sin(90^\circ\pm\theta)$.]

23. If $\operatorname{cosec} A+\sec A=\operatorname{cosec} B+\sec B$, then
$$\tan A+\tan B=\cot\tfrac{1}{2}(A+B).$$ [*P. U. 1936*]

24. Prove that
$$\left(\frac{\cos A+\cos B}{\sin A-\sin B}\right)^n+\left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n=2\cot^n\frac{A-B}{2},$$
or zero, according as n is even or odd. [*P.U. 1933*]

CHAPTER VIII

MULTIPLE ANGLES

43. Trigonometrical ratios of angle 2A.

From Art. 33, we have,

$$\sin(A+B)=\sin A\cos B+\cos A\sin B.$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B.$$

Putting $B=A$, in the first formula, we get

$$\sin 2\mathbf{A}=\sin A\cos A+\cos A\sin A=\mathbf{2\sin A\cos A} \quad (1)$$

Putting $B=A$, in the second formula, we get

$$\cos 2\mathbf{A}=\cos A.\cos A-\sin A.\sin A=\mathbf{\cos^2 A-\sin^2 A} \quad (2)$$

$$=(1-\sin^2 A)-\sin^2 A=\mathbf{1-2\sin^2 A} \quad \cdots (3)$$

and also $=\cos^2 A-(1-\cos^2 A)=\mathbf{2\cos^2 A-1}.$ $\quad\cdots (4)$

By Art. 36, $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}.$

Putting $B=A$, in the above formula, we get

$$\mathbf{\tan 2A}=\frac{\tan A+\tan A}{1-\tan A.\tan A}=\mathbf{\frac{2\tan A}{1-\tan^2 A}}. \quad\cdots (5)$$

Similarly, putting $B=A$ in the value of $\cot(A+B)$ as given in Art. 37, we get $\cot 2A=\dfrac{\cot^2 A-1}{2\cot A}.$ $\quad\cdots (6)$

From formulæ (3) and (4), we obtain, by transposition,

$$\mathbf{1+\cos 2A=2\cos^2 A} \quad\cdots\quad\cdots (7)$$

$$\mathbf{1-\cos 2A=2\sin_2 A} \quad\cdots\quad\cdots (8)$$

By division, $\mathbf{\dfrac{1-\cos 2A}{1+\cos 2A}=\tan^2 A}$ $\quad\cdots\quad\cdots (9)$

We may also note that

$$1+\sin 2A=\cos^2 A+\sin^2 A+2\sin A\cos A$$
$$=(\cos A+\sin A)^2$$
$$1-\sin 2A=\cos^2 A+\sin^2 A-2\sin A\cos A$$
$$=(\cos A-\sin A)^2.$$

Note. Since the addition formulæ are perfectly general (*i.e.*, true for all values of A and B), the above formulæ, being deduced from addition formulæ, are also perfectly general.

44. Trigonometrical ratios of angle 3A.

$$\sin 3A=\sin(A+2A)=\sin A\cos 2A+\cos A\sin 2A$$
$$=\sin A(1-2\sin^2 A)+\cos A\,.\,2\sin A\cos A$$

[*By Art.* 43]

$$=\sin A(1-2\sin^2 A)+2\sin A(1-\sin^2 A).$$

$\therefore$ **$\sin 3A=3\sin A-4\sin^3 A$.**

$$\cos 3A=\cos(A+2A)=\cos A\cos 2A-\sin A\sin 2A$$
$$=\cos A(2\cos^2 A-1)-\sin A\,.\,2\sin A\cos A$$
$$=\cos A(2\cos^2 A-1)-2\cos A\,.\,\sin^2 A$$
$$=2\cos^3 A-\cos A-2\cos A(1-\cos^2 A).$$

$\therefore$ **$\cos 3A=4\cos^3 A-3\cos A$.**

$$\tan 3A=\tan(A+2A)=\frac{\tan A+\tan 2A}{1-\tan A\tan 2A}$$

$$=\frac{\tan A+\dfrac{2\tan A}{1-\tan^2 A}}{1-\tan A\dfrac{2\tan A}{1-\tan^2 A}}$$

$$=\frac{\tan A(1-\tan^2 A)+2\tan A}{(1-\tan^2 A)-2\tan^2 A}.$$

$\therefore$ **$\tan 3A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$.**

Obs. By a method similar to that of the previous article the Trigonometrical ratios of any higher multiple of A can be expressed in terms of those of A.

45. Ex. 1. *Express sin 2A and cos 2A in terms of tan A.*

$$\sin 2A = 2\sin A\cos A = 2\frac{\sin A}{\cos A}\cdot\cos^2 A$$

$$= 2\tan A\frac{1}{\sec^2 A} = \frac{2\tan A}{1+\tan^2 A}.$$

$$\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - \cos^2 A\cdot\frac{\sin^2 A}{\cos^2 A}$$

$$= \cos^2 A\left(1 - \frac{\sin^2 A}{\cos^2 A}\right) = \frac{1}{\sec^2 A}(1-\tan^2 A)$$

$$= \frac{1-\tan^2 A}{1+\tan^2 A}.$$

Ex. 2. *Express cos 4A in terms of cos A.*

Putting $\theta = 2A$, $\cos 4A = \cos 2\theta = 2\cos^2\theta - 1$

$$= 2(\cos 2A)^2 - 1$$

$$= 2(2\cos^2 A - 1)^2 - 1$$

$$= 8\cos^4 A - 8\cos^2 A + 1.$$

Ex. 3. *Show that* $\dfrac{1-\tan^2(45^\circ - A)}{1+\tan^2(45^\circ - A)} = \sin 2A.$

Let $\theta = 45^\circ - A$; then

$$\text{Left side} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\dfrac{\sin^2\theta}{\cos^2\theta}}{1+\dfrac{\sin^2\theta}{\cos^2\theta}} = \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$= \cos(90^\circ - 2A) = \sin 2A.$$

Examples VIII

Prove the following identities (*Ex. 1 to 25*) :—

1. $\dfrac{\sin 2A}{1+\cos 2A}=\tan A.$

2. $\dfrac{\sin 2A}{1-\cos 2A}=\cot A.$

3. $\cot A-\tan A=2\cot 2A.$

4. (i) $(2\cos\theta+1)(2\cos\theta-1)=2\cos 2\theta+1.$

   (ii) $\tan\theta\,(1+\sec 2\theta)=\tan 2\theta.$

5. $\dfrac{\cot A-\tan A}{\cot A+\tan A}=\cos 2A.$

6. $\tan A+\cot A=2\operatorname{cosec} 2A.$

7. $\cos^4\theta-\sin^4\theta=\cos 2\theta.$

8. $\cos^6\theta-\sin^6\theta=\cos 2\theta\,(1-\frac{1}{4}\sin^2 2\theta).$

9. $\cos^6\theta+\sin^6\theta=\frac{1}{4}(1+3\cos^2 2\theta).$

10. $\dfrac{\sin^2\alpha-\sin^2\beta}{\sin\alpha\cos\alpha-\sin\beta\cos\beta}=\tan(\alpha+\beta).$

11. (i) $\dfrac{1-\cos 2\theta+\sin 2\theta}{1+\cos 2\theta+\sin 2\theta}=\tan\theta.$ [*C. U. 1938*]

    (ii) $\dfrac{\sin\alpha-\sqrt{1+\sin 2\alpha}}{\cos\alpha-\sqrt{1+\sin 2\alpha}}=\cot\alpha.$ [α being positive and acute, and the square root being taken with positive sign.]

12. $\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}-\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=2\tan 2\theta.$

13. (i) $\dfrac{\sin A+\sin 2A}{1+\cos A+\cos 2A}=\tan A.$

    (ii) $\dfrac{\sin 4\theta}{\cos 2\theta}\cdot\dfrac{1-\cos 2\theta}{1-\cos 4\theta}=\tan\theta.$

14. (i) $\dfrac{\cos A-\sin A}{\cos A+\sin A}=\sec 2A-\tan 2A.$

    (ii) $\dfrac{\cos^3\theta+\sin^3\theta}{\cos\theta+\sin\theta}=1-\frac{1}{2}\sin 2\theta.$

15. $\cos^3 A \cos 3A + \sin^3 A \sin 3A = \cos^3 2A.$

16. (i) $4(\cos^3 10^\circ + \sin^3 20^\circ) = 3(\cos 10^\circ + \sin 20^\circ).$

(ii) $\dfrac{1}{\sin 10^\circ} - \dfrac{\sqrt{3}}{\cos 10^\circ} = 4.$

17. $\tan 3\theta - \tan 2\theta - \tan \theta = \tan 3\theta \tan 2\theta \tan \theta.$

18. $\dfrac{\cot A}{\cot A - \cot 3A} - \dfrac{\tan A}{\tan 3A - \tan A} = 1.$

19. $\dfrac{1}{\tan 3\theta - \tan \theta} - \dfrac{1}{\cot 3\theta - \cot \theta} = \cot 2\theta.$

20. $\sin 8\theta = 8 \sin \theta \cos \theta \cos 2\theta \cos 4\theta.$

21. (i) $\cos 5\theta = 16 \cos^5\theta - 20 \cos^3\theta + 5 \cos \theta.$

(ii) $\sin 5\theta = 16 \sin^5\theta - 20 \sin^3\theta + 5 \sin \theta.$

22. (i) $\cot 3\theta = \dfrac{\cot^3\theta - 3 \cot \theta}{3 \cot^2\theta - 1}.$

(ii) $\tan 4\theta = \dfrac{4 \tan \theta - 4 \tan^3\theta}{1 - 6 \tan^2\theta + \tan^4\theta}.$

23. (i) $\cos(120^\circ - A) + \cos A + \cos(120^\circ + A) = 0.$

(ii) $\cos^2(A - 120^\circ) + \cos^2 A + \cos^2(A + 120^\circ) = \frac{3}{2}.$

24. $\dfrac{2 \cos 2^n\theta + 1}{2 \cos \theta + 1} = (2 \cos \theta - 1)(2 \cos 2\theta - 1)(2 \cos 2^2\theta - 1).$

$\cdots\cdots(2 \cos 2^{n-1}\theta - 1).$

[Use $(2 \cos \theta + 1)(2 \cos \theta - 1) = 2 \cos 2\theta + 1.$]

25. $\dfrac{\tan 2^n\theta}{\tan \theta} = (1 + \sec 2\theta)(1 + \sec 2^2\theta)(1 + \sec 2^3\theta)\cdots$

$\cdots(1 + \sec 2^n\theta).$

[Use $\tan \theta\,(1 + \sec 2\theta) = \tan 2\theta.$]

26. If $\theta = \dfrac{\pi}{2^n + 1}$, prove that

$$2^n \cos \theta \cos 2\theta \cos 2^2\theta \cdots \cos 2^{n-1}\theta = 1.$$

27. (i) If $\tan x = b/a$, find the value of $a \cos 2x + b \sin 2x$.

(ii) If $\tan^2 x + 2 \tan x \tan 2y = \tan^2 y + 2 \tan y \tan 2x$, prove that each side $= 1$, or, else, $\tan x = \pm \tan y$.

28. If $\tan^2\theta = 1 + 2\tan^2\phi$, show that $\cos 2\phi = 1 + 2\cos 2\theta$.

29. (i) If $2\tan\alpha = 3\tan\beta$, prove that

$$\tan(\alpha - \beta) = \frac{\sin 2\beta}{5 - \cos 2\beta}. \qquad \text{[C. U. 1946]}$$

(ii) If $\dfrac{\tan(\alpha - \beta - \gamma)}{\tan(\alpha - \beta + \gamma)} = \dfrac{\tan\gamma}{\tan\beta}$, show that

either, $\sin(\beta - \gamma) = 0$, *or*, $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$.

30. If α and β are acute angles and $\cos 2\alpha = \dfrac{3\cos 2\beta - 1}{3 - \cos 2\beta}$, show that $\tan\alpha = \sqrt{2}\tan\beta$. [*C. U. 1941*]

31. If $\cos\theta = \frac{1}{2}(a + a^{-1})$, show that

(i) $\cos 2\theta = \frac{1}{2}(a^2 + a^{-2})$.

(ii) $\cos 3\theta = \frac{1}{2}(a^3 + a^{-3})$.

Show that (*Ex. 32 to 36*) :—

32. $\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$.

33. $\cos^8\theta + \sin^8\theta = 1 - \sin^2 2\theta + \frac{1}{8}\sin^4 2\theta$.

34. $\tan\left(\dfrac{\pi}{4} + A\right) + \tan\left(\dfrac{\pi}{4} - A\right) = 2\sec 2A$.

35. $\cos^3\theta\,\dfrac{\sin 3\theta}{3} + \sin^3\theta\,\dfrac{\cos 3\theta}{3} = \dfrac{\sin 4\theta}{4}$.

36. $\cos 4x - \cos 4y$

$$= 8(\cos x - \cos y)(\cos x + \cos y)(\cos x - \sin y)$$
$$\times (\cos x + \sin y).$$

CHAPTER IX

SUBMULTIPLE ANGLES

46. From the usual formulæ for multiple angles, namely

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$

$$1 + \cos 2A = 2 \cos^2 A \ ; \ 1 - \cos 2A = 2 \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

putting $A = \frac{1}{2}\theta$ and $\frac{1}{3}\theta$ respectively we derive the following formulæ for submultiple angles :

$$\sin \theta = 2 \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta$$

$$\cos \theta = \cos^2 \tfrac{1}{2}\theta - \sin^2 \tfrac{1}{2}\theta = 2 \cos^2 \tfrac{1}{2}\theta - 1 = 1 - 2 \sin^2 \tfrac{1}{2}\theta$$

$$1 + \cos \theta = 2 \cos^2 \tfrac{1}{2}\theta \ ; \ 1 - \cos \theta = 2 \sin^2 \tfrac{1}{2}\theta$$

$$\tan \theta = \frac{2 \tan \frac{1}{2}\theta}{1 - \tan^2 \frac{1}{2}\theta}$$

$$\sin \theta = 3 \sin \tfrac{1}{3}\theta - 4 \sin^3 \tfrac{1}{3}\theta$$

$$\cos \theta = 4 \cos^3 \tfrac{1}{3}\theta - 3 \cos \tfrac{1}{3}\theta$$

$$\tan \theta = \frac{3 \tan \frac{1}{3}\theta - \tan^3 \frac{1}{3}\theta}{1 - 3 \tan^2 \frac{1}{3}\theta}.$$

47. Values of $\sin \frac{1}{2}\theta$ and $\cos \frac{1}{2}\theta$ in terms of $\cos \theta$.

From $\cos \theta = 2 \cos^2 \frac{1}{2}\theta - 1 = 1 - 2 \sin^2 \frac{1}{2}\theta$, we at once deduce

$$\sin \tfrac{1}{2}\theta = \pm \sqrt{\tfrac{1}{2}(1 - \cos \theta)}$$

$$\cos \tfrac{1}{2}\theta = \pm \sqrt{\tfrac{1}{2}(1 + \cos \theta)}$$

48. *Ambiguity of signs explained.*

When $\cos\theta$ is given and not θ, θ and consequently $\frac{1}{2}\theta$ has a series of values as will be explained in Chapter XI. Thus, $\frac{1}{2}\theta$ may lie in any quadrant and $\sin\frac{1}{2}\theta$ and $\cos\frac{1}{2}\theta$ will then have corresponding signs.

If the quadrant in which $\frac{1}{2}\theta$ lies be known, for example, when θ is given along with $\cos\theta$, there is no ambiguity in choosing the proper signs of $\cos\frac{1}{2}\theta$ and $\sin\frac{1}{2}\theta$, as shown in the following example.

Ex. *Find* $\sin 22\frac{1}{2}°$ *and* $\cos 22\frac{1}{2}°$.

$$\sin 22\tfrac{1}{2}° = +\sqrt{\tfrac{1}{2}(1-\cos 45°)} = \sqrt{\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)} = \frac{1}{2}\sqrt{2-\sqrt{2}}$$

$$\cos 22\tfrac{1}{2}° = +\sqrt{\tfrac{1}{2}(1+\cos 45°)} = \sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)} = \frac{1}{2}\sqrt{2+\sqrt{2}}$$

49. Values of $\sin\frac{1}{2}\theta$ and $\cos\frac{1}{2}\theta$ in terms of $\sin\theta$.

We know that $\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$

and $\qquad 1 = \cos^2\frac{1}{2}\theta + \sin^2\frac{1}{2}\theta$.

Therefore, $1+\sin\theta = (\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta)^2$,

and $1-\sin\theta = (\cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta)^2$.

Hence, $\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta = \pm\sqrt{1+\sin\theta}$

$\cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta = \pm\sqrt{1-\sin\theta}$.

Thus, $\boldsymbol{\cos\frac{1}{2}\theta = \pm\frac{1}{2}\sqrt{1+\sin\theta} \pm \frac{1}{2}\sqrt{1-\sin\theta}}$

and $\boldsymbol{\sin\frac{1}{2}\theta = \pm\frac{1}{2}\sqrt{1+\sin\theta} \mp \frac{1}{2}\sqrt{1-\sin\theta}}$.

50. *Ambiguity of signs explained.*

As before, when $\sin\theta$ is given, and not θ, θ has a series of values for the given value of $\sin\theta$ as will be explained in Chapter XI; $\frac{1}{2}\theta$ may therefore lie in any one of two possible quadrants.

$$\cos \tfrac{1}{2}\theta + \sin \tfrac{1}{2}\theta = \sqrt{2} \sin (\tfrac{1}{4}\pi + \tfrac{1}{2}\theta)$$

and $\cos \tfrac{1}{2}\theta - \sin \tfrac{1}{2}\theta = \sqrt{2} \sin (\tfrac{1}{4}\pi - \tfrac{1}{2}\theta)$

will have their signs determined accordingly.

Thus, $\sin \tfrac{1}{2}\theta$ and $\cos \tfrac{1}{2}\theta$ will be definitely known.

Ex. *Find* $\sin 15°$ *and* $\cos 15°$.

We have, $\cos 15° + \sin 15° = + \sqrt{1 + \sin 30°} = \sqrt{1 + \tfrac{1}{2}}$

$\cos 15° - \sin 15° = + \sqrt{1 - \sin 30°} = \sqrt{1 - \tfrac{1}{2}}$.

[$\cos 15° - \sin 15° = \sqrt{2} \sin (\tfrac{1}{4}\pi - 15°)$ and is clearly positive.]

Thus, $\cos 15° = \tfrac{1}{2} \left(\sqrt{\tfrac{3}{2}} + \sqrt{\tfrac{1}{2}} \right) = \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$

$\sin 15° = \tfrac{1}{2} \left(\sqrt{\tfrac{3}{2}} - \sqrt{\tfrac{1}{2}} \right) = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$.

51. tan $\tfrac{1}{2}\theta$ in terms of tan θ.

From the formula, $\tan \theta = \dfrac{2 \tan \tfrac{1}{2}\theta}{1 - \tan^2 \tfrac{1}{2}\theta}$,

i.e., $\tan \theta \tan^2 \tfrac{1}{2}\theta + 2 \tan \tfrac{1}{2}\theta - \tan \theta = 0$,

we easily deduce

$$\tan \tfrac{1}{2}\theta = \frac{-1 \pm \sqrt{1 + \tan^2 \theta}}{\tan \theta}.$$

The reason of the ambiguity is similar to those of the previous cases.

52. Ratios of $\tfrac{1}{3}\theta$ from those of θ.

By solving the cubic equation

$$\sin \theta = 3 \sin \tfrac{1}{3}\theta - 4 \sin^3 \tfrac{1}{3}\theta \quad \cdots \quad \cdots \quad (1)$$

we get $\sin \tfrac{1}{3}\theta$, if $\sin \theta$ be known.

Similarly, by solving the cubic equations

$$\cos \theta = 4 \cos^3 \tfrac{1}{3}\theta - 3 \cos \tfrac{1}{3}\theta \quad \cdots \quad \cdots \quad (2)$$

and $$\tan\theta = \frac{3\tan\frac{1}{3}\theta - \tan^3\frac{1}{3}\theta}{1 - 3\tan^2\frac{1}{3}\theta} \quad \cdots \quad (3)$$

we derive values of $\cos\frac{1}{3}\theta$ from those of $\cos\theta$, and of $\tan\frac{1}{3}\theta$ from those of $\tan\theta$ respectively.

53. Ratios of 18° and 36°.

Let $\theta = 18°$; then $5\theta = 90°$; $\therefore\ 2\theta = 90° - 3\theta$.

$\therefore\ \sin 2\theta = \cos 3\theta$, or, $2\sin\theta\cos\theta = \cos\theta\,(4\cos^2\theta - 3)$.

As $\cos\theta$ (*i.e.*, $\cos 18°$) is not zero, we have

$$2\sin\theta = 4\cos^2\theta - 3 = 1 - 4\sin^2\theta,$$

or $$4\sin^2\theta + 2\sin\theta - 1 = 0.$$

$$\therefore\ \sin\theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{(\pm\sqrt{5} - 1)}{4}.$$

Now, as θ here is a positive acute angle, therefore, rejecting the negative value, we get

$$\mathbf{\sin 18° = \tfrac{1}{4}(\sqrt{5} - 1).}$$
$$\cos 18° = +\sqrt{1 - \sin^2 18°} = \tfrac{1}{4}\sqrt{(10 + 2\sqrt{5})}.$$
$$\cos 36° = 1 - 2\sin^2 18° = \tfrac{1}{4}(\sqrt{5} + 1).$$
$$\sin 36° = \sqrt{1 - \cos^2 36°} = \tfrac{1}{4}\sqrt{(10 - 2\sqrt{5})}.$$

Note. Since 54° and 36° are complementary and 72° and 18° are complementary, from the above values we easily get the trigonometrical ratios of 54° and 72°.

54. Ratios of 3° and multiples of 3°.

$$\sin 3° = \sin(18° - 15°) = \sin 18°\cos 15° - \cos 18°\sin 15°$$
$$= \tfrac{1}{16}(\sqrt{5} - 1)(\sqrt{6} + \sqrt{2}) - \tfrac{1}{8}(\sqrt{3} - 1)\sqrt{(5 + \sqrt{5})},$$

on substituting the values of $\sin 18°$, $\cos 15°$, etc.

Similarly,

$$\cos 3° = \tfrac{1}{8}(\sqrt{3} + 1)\sqrt{(5 + \sqrt{5})} + \tfrac{1}{16}(\sqrt{6} - \sqrt{2})(\sqrt{5} - 1).$$

From a knowledge of the ratios of 3°, 15°, 18°, 30°, 36° and 45°, we can deduce the ratios for all angles which

are multiples of $3°$, (for, $6° = 36° - 30°$; $9° = 45° - 36°$; $12° = 30° - 18°$; $21° = 36° - 15°$; etc.). For angles greater than $45°$, the ratios may be deduced from those of their complements which are less than $45°$.

Ex. *Show that*

$$\sin x = 2^n \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cdots\cdots\cos\frac{x}{2^n}\sin\frac{x}{2^n}.$$

We have, $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$$\sin\frac{x}{2} = 2\sin\frac{x}{2^2}\cos\frac{x}{2^2}$$

$$\sin\frac{x}{2^2} = 2\sin\frac{x}{2^3}\cos\frac{x}{2^3}$$

$$\cdots \qquad \cdots \qquad \cdots$$

Similarly, $\sin\frac{x}{2^{n-1}} = 2\sin\frac{x}{2^n}\cos\frac{x}{2^n}.$

Hence, $\sin x = 2^n\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cdots\cdots\cos\frac{x}{2^n}\sin\frac{x}{2^n}.$

Examples IX

Prove that (*Ex. 1 to 14*) :—

1. $\dfrac{1-\cos A}{\sin A} = \tan\dfrac{A}{2}$. **2.** $\dfrac{1+\cos A}{\sin A} = \cot\dfrac{A}{2}$.

3. $\left(\sin\dfrac{A}{2} \pm \cos\dfrac{A}{2}\right)^2 = 1 \pm \sin A$.

4. $\sec\theta + \tan\theta = \tan(\frac{1}{4}\pi + \frac{1}{2}\theta)$. [*C. U. 1939*]

5. (i) $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\dfrac{\theta}{2}$.

(ii) $\dfrac{\sin\frac{1}{2}\alpha - \sqrt{1+\sin\alpha}}{\cos\frac{1}{2}\alpha - \sqrt{1+\sin\alpha}} = \cot\dfrac{\alpha}{2}$ where $0 < \alpha < \pi$,

and the square root is taken with positive sign.

6. (i) $\dfrac{1+\sin x}{1-\sin x} = \tan^2\left(\dfrac{\pi}{2} + \dfrac{x}{2}\right)$.

(ii) $\dfrac{2\sin\theta - \sin 2\theta}{2\sin\theta + \sin 2\theta} = \tan^2\frac{1}{2}\theta$.

7. (i) $\dfrac{1+\tan\frac{1}{2}A}{1-\tan\frac{1}{2}A}=\dfrac{1+\sin A}{\cos A}$.

(ii) $\cot\beta=\frac{1}{2}(\cot\frac{1}{2}\beta-\tan\frac{1}{2}\beta)$.

8. (i) $\dfrac{\sin 2\theta}{1+\cos 2\theta}\cdot\dfrac{\cos\theta}{1+\cos\theta}=\tan\dfrac{\theta}{2}$.

(ii) $8\sin^4\frac{1}{2}\theta-8\sin^2\frac{1}{2}\theta+1=\cos 2\theta$.

9. $\sin\theta=\dfrac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$. **10.** $\cos\theta=\dfrac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$.

11. $(\cos x+\cos y)^2+(\sin x+\sin y)^2=4\cos^2\frac{1}{2}(x-y)$.

12. $\tan 6^\circ\tan 42^\circ\tan 66^\circ\tan 78^\circ=1$.

13. $\tan 7\frac{1}{2}^\circ=\sqrt{6}-\sqrt{3}+\sqrt{2}-2$.

14. $2\cos\frac{1}{16}\pi=\sqrt{2+\sqrt{2+\sqrt{2}}}$.

15. (i) If $\tan\dfrac{\theta}{2}=\sqrt{\dfrac{1-e}{1+e}}\cdot\tan\dfrac{\phi}{2}$, show that

$$\cos\phi=\frac{\cos\theta-e}{1-e\cos\theta}.$$

(ii) If $\tan\theta=\dfrac{\sin\alpha\sin\beta}{\cos\alpha+\cos\beta}$, show that one of the values of $\tan\frac{1}{2}\theta$ is $\tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta$.

16. If $\sin\alpha+\sin\beta=a$ and $\cos\alpha+\cos\beta=b$, find the value of $\cos(\alpha+\beta)$.

17. (i) Prove that $2\sin\frac{1}{2}A=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}$, and determine which are the correct signs when $270^\circ>A>180^\circ$ [*B. H. U. I., 1931*]

(ii) If $\theta=240^\circ$, is the following statement correct ?

$$2\sin\tfrac{1}{2}\theta=\sqrt{1+\sin\theta}-\sqrt{1-\sin\theta}.$$

If not, how must it be modified ?

18. If $A=320^\circ$, prove that

$$\tan\frac{A}{2}=\frac{-1+\sqrt{1+\tan^2 A}}{\tan A}.$$

CHAPTER X

TRIGONOMETRICAL IDENTITIES

55. Many interesting identities involving functions of three or more angles can be established when there exists a relation among the angles. The most important of these identities are those in which the three angles are connected by the relation that their sum is equal to two right angles. In establishing this latter kind of identities, it will be necessary to make frequent use of the properties of supplementary and complementary angles.

Thus, since $A+B+C=\pi$,

$$\therefore \quad B+C=\pi-A.$$

$$\therefore \quad \sin(B+C)=\sin(\pi-A)=\sin A.$$

Similarly, $\sin(C+A)=\sin B$; $\sin(A+B)=\sin C$.

Again, $\cos(B+C)=\cos(\pi-A)=-\cos A$.

Similarly, $\cos(C+A)=-\cos B$; $\cos(A+B)=-\cos C$.

$$\tan(B+C)=\tan(\pi-A)=-\tan A.$$

Similarly, $\tan(C+A)=-\tan B$; $\tan(A+B)=-\tan C$.

Again, since, $\dfrac{A}{2}+\dfrac{B}{2}+\dfrac{C}{2}=\dfrac{\pi}{2}$,

$$\therefore \quad \sin\left(\frac{B}{2}+\frac{C}{2}\right)=\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)=\cos\frac{A}{2}.$$

Similarly,
$$\sin\left(\frac{C}{2}+\frac{A}{2}\right)=\cos\frac{B}{2};$$

$$\sin\left(\frac{A}{2}+\frac{B}{2}\right)=\cos\frac{C}{2}.$$

Again,
$$\cos\left(\frac{B}{2}+\frac{C}{2}\right)=\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)=\sin\frac{A}{2}.$$

Similarly, $\cos\left(\frac{C}{2}+\frac{A}{2}\right)=\sin\frac{B}{2}$;

$$\cos\left(\frac{A}{2}+\frac{B}{2}\right)=\sin\frac{C}{2}.$$

$$\tan\left(\frac{B}{2}+\frac{C}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{A}{2}\right)=\cot\frac{A}{2}.$$

Similarly, $\tan\left(\frac{C}{2}+\frac{A}{2}\right)=\cot\frac{B}{2}$;

$$\tan\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\frac{C}{2}.$$

56. Ex. 1. ***If* $A+B+C=\pi$, *prove that***
$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$.

[*C. U. 1931, '33, '35, H. S. '61 Comp.*]

Left side $=(\sin 2A+\sin 2B)+\sin 2C$

$=2\sin(A+B)\cos(A-B)+2\sin C\cos C$

$=2\sin C\cos(A-B)+2\sin C\cos C$ [$\because A+B+C=\pi$.]

$=2\sin C[\cos(A-B)+\cos C]$

$=2\sin C[\cos(A-B)-\cos(A+B)]$ [$\because A+B+C=\pi$.]

$=2\sin C.2\sin A\sin B$

$=4\sin A\sin B\sin C$.

Ex. 2. ***If* $A+B+C=\pi$, *prove that***
$\cos 2A+\cos 2B+\cos 2C=-4\cos A\cos B\cos C-1$.

Left side $=(\cos 2A+\cos 2B)+\cos 2C$

$=2\cos(A+B)\cos(A-B)+2\cos^2C-1$

$=-2\cos C\cos(A-B)+2\cos^2C-1$ [$\because A+B+C=\pi$.]

$=-2\cos C[\cos(A-B)-\cos C]-1$

$=-2\cos C[\cos(A-B)+\cos(A+B)]-1$ [$\because A+B+C=\pi$.]

$=-2\cos C.2\cos A\cos B-1$

$=-4\cos A\cos B\cos C-1$.

Ex. 3. *If* $A+B+C=\pi$, *prove that*

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

[*C. U. 1910, '29, H. S. '60 Comp.*]

Left side $=(\sin A+\sin B)+\sin C$

$$=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$=2\cos\frac{C}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$\left[\because \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}.\right]$$

$$=2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2}+\sin\frac{C}{2}\right\}$$

$$=2\cos\frac{C}{2}\left[\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right]$$

$$\left[\because \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}.\right]$$

$$=2\cos\frac{C}{2}\cdot 2\cos\frac{A}{2}\cos\frac{B}{2}$$

$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Ex. 4. *If* $A+B+C=\pi$, *prove that*

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Left side $=(\cos A+\cos B)+\cos C$

$$=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}$$

$$=2\sin\frac{C}{2}\cos\frac{A-B}{2}-2\sin^2\frac{C}{2}+1$$

$$\left[\because \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}.\right]$$

$$=2\sin\frac{C}{2}\left[\cos\frac{A-B}{2}-\sin\frac{C}{2}\right]+1$$

$$=2\sin\frac{C}{2}\left[\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right]+1$$

$$\left[\because\ \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}.\right]$$

$$=2\sin\frac{C}{2}\cdot 2\sin\frac{A}{2}\sin\frac{B}{2}+1$$

$$=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Ex. 5. ***If*** **$A+B+C=\pi$,** ***prove that***

$\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

[*H. S. 1961*]

Since, $B+C=\pi-A$,

$\therefore\quad \tan(B+C)=\tan(\pi-A)$.

$$\therefore\quad \frac{\tan B+\tan C}{1-\tan B\tan C}=-\tan A,$$

i.e., $\tan B+\tan C=-\tan A\,(1-\tan B\tan C)$

$\qquad\qquad\qquad\qquad =-\tan A+\tan A\tan B\tan C$.

$\therefore\quad \tan A+\tan B+\tan C=\tan A\tan B\tan C$.

Otherwise :

$\tan(A+B+C)=\tan\pi=0$.

$$\therefore\quad \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan B\tan C-\tan C\tan A-\tan A\tan B}=0.$$

Since, the fraction is zero, numerator must be zero.

$\therefore\quad \tan A+\tan B+\tan C-\tan A\tan B\tan C=0$,

i.e., $\tan A+\tan A+\tan C=\tan A\tan B\tan C$.

Ex. 6. ***If*** **$A+B+C=\pi$,** ***prove that***

$$\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1.$$

[*C. U. 1936, '39*]

Since, $A+B+C=\pi$, $\therefore$ $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$.

$$\therefore \quad \tan\left(\frac{B}{2}+\frac{C}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{A}{2}\right).$$

$$\therefore \quad \frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{1-\tan\frac{B}{2}\tan\frac{C}{2}}=\cot\frac{A}{2}=\frac{1}{\tan\frac{A}{2}}$$

or, $\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=1-\tan\frac{B}{2}\tan\frac{C}{2}$.

On simplification, the required result follows.

Otherwise .

$$\cdot\ 1\Big/\tan\left(\frac{A}{2}+\frac{B}{2}+\frac{C}{2}\right)=\cot\left(\frac{A}{2}+\frac{B}{2}+\frac{C}{2}\right)=\cot\frac{\pi}{2}.$$

$$\therefore \quad \frac{1-\tan\frac{B}{2}\tan\frac{C}{2}-\tan\frac{C}{2}\tan\frac{A}{2}-\tan\frac{A}{2}\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}-\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}=0.$$

Now the value of the fraction being zero, its numerator must be zero.

$$\therefore \quad 1-\tan\frac{B}{2}\tan\frac{C}{2}-\tan\frac{C}{2}\tan\frac{A}{2}-\tan\frac{A}{2}\tan\frac{B}{2}=0,$$

whence the required result follows.

Ex. 7. ***If* A+B+C=π, *prove that***

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}.$$

Right side $=2\cos\frac{\pi-A}{4}\left[2\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}\right]$

$$=2\cos\frac{\pi-A}{4}\left[\cos\frac{2\pi-(B+C)}{4}+\cos\frac{B-C}{4}\right]$$

$$=2\cos\frac{\pi-A}{4}\left[\cos\frac{\pi+A}{4}+\cos\frac{B-C}{4}\right]$$

[$\because$ $2\pi-(B+C)=\pi+\pi-(B+C)=\pi+A$, since, $A+B+C=\pi$.]

$$= 2\cos\frac{\pi - A}{4}\cos\frac{\pi + A}{4} + 2\cos\frac{\pi - A}{4}\cos\frac{B - C}{4}$$

$$= \left(\cos\frac{\pi}{2} + \cos\frac{A}{2}\right) + 2\cos\frac{B+C}{4}\cos\frac{B-C}{4}$$

[$\because$ $A+B+C=\pi$.]

$$= \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}.$$

Note. Since, $\cos\frac{1}{4}(\pi - A) = \sin\{\frac{1}{2}\pi - \frac{1}{4}(\pi - A)\} = \sin\frac{1}{4}(\pi + A)$
and $\cos\frac{1}{4}(\pi - A) = \cos\frac{1}{4}(A + B + C - A) = \cos\frac{1}{4}(B + C)$,

$\therefore$ we have also, $\cos\frac{1}{2}A + \cos\frac{1}{2}B + \cos\frac{1}{2}C$

$= 4\sin\frac{1}{4}(\pi + A)\sin\frac{1}{4}(\pi + B)\sin\frac{1}{4}(\pi + C)$

$= 4\cos\frac{1}{4}(B + C)\cos\frac{1}{4}(C + A)\cos\frac{1}{4}(A + B)$.

Ex. 8. ***If*** $\mathbf{A+B+C=\pi}$, ***prove that***

$$\mathbf{\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1.}$$

[*C. U. 1932, '37, '47*]

$\cos^2 A + \cos^2 B + \cos^2 C$

$$= \tfrac{1}{2}(2\cos^2 A + 2\cos^2 B) + \cos^2 C$$

$$= \tfrac{1}{2}(1 + \cos 2A + 1 + \cos 2B) + \cos^2 C$$

$$= 1 + \tfrac{1}{2}(\cos 2A + \cos 2B) + \cos^2 C$$

$$= 1 + \cos(A + B)\cos(A - B) + \cos C \,.\, \cos C$$

$$= 1 - \cos C\cos(A - B) - \cos C\cos(A + B)$$

[$\because$ $A+B=\pi-C$.]

$$= 1 - \cos C\,[\,\cos(A - B) + \cos(A + B)\,]$$

$$= 1 - \cos C\,[\,2\cos A\cos B\,]$$

$$= 1 - 2\cos A\cos B\cos C,$$

whence the required result follows.

Ex. 9. *Show that*

$$\tan(\beta - \gamma) + \tan(\gamma - \alpha) + \tan(\alpha - \beta)$$

$$= \tan(\beta - \gamma)\tan(\gamma - \alpha)\tan(\alpha - \beta).$$

Let $A=\beta-\gamma,\ B=\gamma-\alpha,\ C=\alpha-\beta$;

then $A+B+C=\beta-\gamma+\gamma-\alpha+\alpha-\beta=0.$

$\therefore$ $\tan(A+B+C)=\tan 0=0.$

$\therefore$ $\tan A+\tan B+\tan C=\tan A\tan B\tan C.$

Now, substituting the values for A, B, C, the required result follows.

Ex. 10. *If $x+y+z=xyz$, prove that*

$$x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)=4xyz.$$

Putting $x=\tan\alpha$, $y=\tan\beta$, $z=\tan\gamma$, in the given relation, we have

$$\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma.$$

$\therefore$ by transposition,

$$\tan\alpha\,(1-\tan\beta\tan\gamma)=-(\tan\beta+\tan\gamma),$$

i.e., $\tan\alpha=-\dfrac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}=-\tan(\beta+\gamma).$

$\therefore$ $\alpha=\pi-(\beta+\gamma)$. $\therefore$ $\alpha+\beta+\gamma=\pi$. $\therefore$ $2\alpha+2\beta+2\gamma=2\pi$.

$\therefore$ $\tan(2\alpha+2\beta+2\gamma)=\tan 2\pi=0.$

Therefore, as in Ex. 5 above,

$$\tan 2\alpha+\tan 2\beta+\tan 2\gamma=\tan 2\alpha\tan 2\beta\tan 2\gamma.$$

Now, expressing $\tan 2\alpha$, $\tan 2\beta$, $\tan 2\gamma$ in terms of $\tan\alpha$, $\tan\beta$, $\tan\gamma$ and substituting x, y, z, for them, we get,

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

On simplification, the required result follows.

Examples X

If $A+B+C=\pi$, prove that (*Ex. 1 to 16*) .—

1. $\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$

[*H. S. 1962*]

2. $\cot B\cot C+\cot C\cot A+\cot A\cot B=1.$

3. $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$.

4. $\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$.

5. $(\cot B + \cot C)(\cot C + \cot A)(\cot A + \cot B)$
$= \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$.

6. $\frac{\cot B + \cot C}{\tan B + \tan C} + \frac{\cot C + \cot A}{\tan C + \tan A} + \frac{\cot A + \cot B}{\tan A + \tan B} = 1$.

7. $\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2}$.
$= 1 + 4 \sin \frac{\pi - A}{4} \sin \frac{\pi - B}{4} \sin \frac{\pi - C}{4}$
$= 1 + 4 \sin \frac{B + C}{4} \sin \frac{C + A}{4} \sin \frac{A + B}{4}$.

8. $\cos^2 2A + \cos^2 2B + \cos^2 2C$
$= 1 + 2 \cos 2A \cos 2B \cos 2C$.

9. $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$.

10. $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$.

11. $\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2$.

[C. U. 1949]

12. $\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$.

13. $\sin (B + C - A) + \sin (C + A - B) + \sin (A + B - C)$
$= 4 \sin A \sin B \sin C$.

14. $\sin (B + 2C) + \sin (C + 2A) + \sin (A + 2B)$
$= 4 \sin \frac{B - C}{2} \sin \frac{C - A}{2} \sin \frac{A - B}{2}$.

15. $\cos^2 A + \cos^2 B + 2 \cos A \cos B \cos C = \sin^2 C$.

16. $\cos \frac{A}{2} \cos \frac{B - C}{2} + \cos \frac{B}{2} \cos \frac{C - A}{2} + \cos \frac{C}{2} \cos \frac{A - B}{2}$.
$= \sin A + \sin B + \sin C$.

17. If $\alpha+\beta+\gamma=\frac{1}{2}\pi$, prove that

(i) $\sin^2\alpha+\sin^2\beta+\sin^2\gamma+2\sin\alpha\sin\beta\sin\gamma=1.$

[*C. U. 1943*]

(ii) $\tan\beta\tan\gamma+\tan\gamma\tan\alpha+\tan\alpha\tan\beta=1.$

18. If A, B, C, D are the angles of a quadrilateral, prove that

(i) $$\frac{\tan A+\tan B+\tan C+\tan D}{\cot A+\cot B+\cot C+\cot D}=\tan A\tan B\tan C\tan D.$$

(ii) $$\cos A+\cos B+\cos C+\cos D=4\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(B+C)\cos\tfrac{1}{2}(C+A).$$

19. Show that

(i) $$\cos^2(\beta-\gamma)+\cos^2(\gamma-\alpha)+\cos^2(\alpha-\beta)=1+2\cos(\beta-\gamma)\cos(\gamma-\alpha)\cos(\alpha-\beta).$$

(ii) $\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta\cos(\alpha+\beta)=\sin^2(\alpha+\beta).$

(iii) $\cos^2\theta+\cos^2(\alpha+\theta)-2\cos\alpha\cos\theta\cos(\alpha+\theta)$ is independent of θ.

20. (i) If $\alpha+\beta=\gamma$, show that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1+2\cos\alpha\cos\beta\cos\gamma.$$

[*C. U. 1940*]

(ii) If $\alpha+\beta+\gamma=2\pi$, show that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma-2\cos\alpha\cos\beta\cos\gamma=1.$$

21. If $\cos(A+B)\sin(C+D)=\cos(A-B)\sin(C-D)$, show that

$$\cot A\cot B\cot C=\cot D.$$

22. If $A+B+C=2S$, prove that

(i) $$\sin(S-A)+\sin(S-B)+\sin(S-C)-\sin S=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

(ii) $$\cos^2A+\cos^2B+\cos^2C+2\cos A\cos B\cos C-1=4\cos S\cos(S-A)\cos(S-B)\cos(S-C).$$

23. If $A+B+C=n\pi$ (n being zero or an integer),

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

24. Show that, if $\alpha+\beta+\gamma=\pi$,

$$\tan(\beta+\gamma-\alpha)+\tan(\gamma+\alpha-\beta)+\tan(\alpha+\beta-\gamma)$$
$$=\tan(\beta+\gamma-\alpha)\tan(\gamma+\alpha-\beta)\tan(\alpha+\beta-\gamma).$$

25. If $A+B+C=\pi$, prove that

(i) $\sin A\cos B\cos C+\sin B\cos C\cos A$
$$+\sin C\cos A\cos B=\sin A\sin B\sin C.$$

(ii) $\cos A\sin B\sin C+\cos B\sin C\sin A$
$$+\cos C\sin A\sin B=1+\cos A\cos B\cos C.$$

(iii) $\sin 5A+\sin 5B+\sin 5C$
$$=4\cos\frac{5A}{2}\cos\frac{5B}{2}\cos\frac{5C}{2}.$$

(iv) $(\tan A+\tan B+\tan C)(\cot A+\cot B+\cot C)$
$$=1+\sec A\sec B\sec C.$$

26. If $\cos A+\cos B+\cos C=0$, show that

$$\cos 3A+\cos 3B+\cos 3C=12\cos A\cos B\cos C.$$

[Write $\cos 3A=4\cos^3 A-3\cos A$, *etc.*]

27. If $x+y+z=\frac{1}{2}\pi$, prove that

$$\cos(x-y-z)+\cos(y-z-x)+\cos(z-x-y)$$
$$-4\cos x\cos y\cos z=0.$$

28. Show that

$$\sin(y-z)+\sin(z-x)+\sin(x-y)$$
$$+4\sin\frac{y-z}{2}\sin\frac{z-x}{2}\sin\frac{x-y}{2}=0.$$

29. If $x+y+z=0$, show that

$$\cot(z+x-y)\cot(x+y-z)+\cot(x+y-z)\cot(y+z-x)$$
$$+\cot(y+z-x)\cot(z+x-y)=1.$$

30. If $x+y+z=xyz$, prove that

$$\frac{3x-x^3}{1-3x^2}+\frac{3y-y^3}{1-3y^2}+\frac{3z-z^3}{1-3z^2}=\frac{3x-x^3}{1-3x^2}\cdot\frac{3y-y^3}{1-3y^2}\cdot\frac{3z-z^3}{1-3z^2}.$$

CHAPTER XI

TRIGONOMETRICAL EQUATIONS AND GENERAL VALUES

57. It will be apparent from Chapter IV that there are infinitely many angles, the trigonometrical ratios of which have a given value. For example, if $\sin\theta = \frac{1}{2}$, one value of θ (the smallest positive value) is known to be $30°$. Now, sines of supplementary angles are equal. Hence, $\sin 150°$ being equal to $\sin 30°$ is also $\frac{1}{2}$. Again, angles differing from $30°$ or $150°$ by complete multiples of $360°$ will have their sines (in fact all ratios) the same. Thus, sine of each of the angles $30°$, $150°$, $390°$, $510°$, $-330°$, $-210°$, etc. is equal to $\frac{1}{2}$. Similarly, if $\cos\theta$ be given, equal to $\frac{1}{\sqrt{2}}$ say, θ may have any of the values $+45°$, $+315°$, $+405°$, $-315°$, $-45°$, etc., or else, if $\tan\theta = \sqrt{3}$, θ may have any of the values $60°$, $240°$, $420°$, $-300°$, etc.

It is very convenient for the solution of trigonometrical equations, as also for other purposes, to obtain a general expression in a compact form embracing all angles, the trigonometrical ratios of which have a given value.

58. General expression of all angles, one of whose trigonometrical ratios is zero.

If the sine of an angle be zero, from definition, the length of the perpendicular from any point of one of its arms upon another is zero, so that the two arms must be in the same straight line. Evidently, therefore, such angles must be zero, or some multiple of π, odd or even.

Thus, if $\mathbf{\sin\theta = 0}$, then $\boldsymbol{\theta = n\pi}$,

n being zero, or any integer, positive or negative.

When the cosine of an angle is zero, the projection of any length along one arm upon another is zero, and so the two arms must be at right angles to one another. The angles must therefore be evidently either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ or differ from these by complete revolutions; in other words, the angle may be any odd multiple of $\frac{\pi}{2}$.

Thus, if $\cos\theta=0$, then $\theta=(2n+1)\frac{\pi}{2}$,

n being zero, or any integer, positive or negative.

Again, if $\tan\theta=0$, then its numerator $\sin\theta$ is also zero;

and so $\theta=n\pi$.

Similarly, if $\cot\theta=0$, then $\cos\theta=0$;

and so $\theta=(2n+1)\frac{\pi}{2}$.

Note. The ratios cosec θ or sec θ can never be zero, for they can never be numerically less than unity.

59. General expression of angles having the same sine *(or cosecant).*

Let α be any angle positive or negative such that its sine is equal to a given quantity k (numerically not greater than 1); for fixing up the idea, and for the sake of convenience in practice, the smallest positive angle having its sine for the given quantity k is taken as α. Let θ be any other angle whose sine is equal to k.

Then, $\sin\theta=\sin\alpha$,

or, $\sin\theta-\sin\alpha=0$,

or, $2\sin\frac{1}{2}(\theta-\alpha)\cos\frac{1}{2}(\theta+\alpha)=0$.

$\therefore$ either $\sin\frac{1}{2}(\theta-\alpha)=0$,

i.e., $\frac{1}{2}(\theta-\alpha)=$ any multiple of $\pi=m\pi$, $\cdots$ (1)

or, else $\cos \frac{1}{2}(\theta + \alpha) = 0$,

i.e., $\frac{1}{2}(\theta + \alpha) =$ any odd multiple of $\frac{\pi}{2} = (2m+1)\frac{\pi}{2}$. $\cdots$ (2)

From (1), $\theta - \alpha = 2m\pi$, *i.e.*, $\theta = \alpha + 2m\pi$. $\cdots$ (3)

From (2), $\theta + \alpha = (2m+1)\pi$, *i.e.*, $\theta = -\alpha + (2m+1)\pi$ $\cdots$ (4)

Combining (3) and (4), $\theta = (-1)^n\alpha + n\pi$ $\cdots$ (5)

where n is zero, or any integer, positive or negative, odd or even.

If $\operatorname{cosec} \theta = \operatorname{cosec} \alpha$, then $\sin \theta = \sin \alpha$; hence all angles having the same cosecant as that of α are also given by the expression (5).

Thus, *all angles having the same sine or cosecant as that of* α *are given by* $2n\pi + \alpha$ and $(2n+1)\pi - \alpha$,

or, $\boldsymbol{n\pi + (-1)^n\alpha}$.

60. General expression of angles having the same cosine *(or secant).*

Let α be the smallest positive angle such that its cosine is equal to a given quantity k (numerically $\not> 1$) ; and let θ be any other angle whose cosine is equal to k.

Then, $\cos \theta = \cos \alpha$,

or, $\cos \alpha - \cos \theta = 0$,

$\therefore$ $2 \sin \frac{1}{2}(\theta + \alpha) \sin \frac{1}{2}(\theta - \alpha) = 0$.

$\therefore$ either $\sin \frac{1}{2}(\theta + \alpha) = 0$,

i.e., $\frac{1}{2}(\theta + \alpha) =$ any multiple of $\pi = n\pi$ $\cdots$ (1)

or else, $\sin \frac{1}{2}(\theta - \alpha) = 0$,

i.e., $\frac{1}{2}(\theta - \alpha) =$ any multiple of $\pi = n\pi$. $\cdots$ (2)

From (1), $\theta + \alpha = 2n\pi$, or $\theta = 2n\pi - \alpha$. $\cdots$ (3)

From (2), $\theta - \alpha = 2n\pi$, or $\theta = 2n\pi + \alpha$. $\cdots$ (4)

From (3) and (4), we have $\theta = 2n\pi \pm \alpha$, ... (5)

where n is zero, or any integer, positive or negative.

It is also evident as in the previous case that all angles having the same secant as that of α are also included in the expression (5).

Hence, *all angles having the same cosine or secant as that of α are given by*

$$2n\pi \pm \alpha$$

n being zero, or any integer, positive or negative.

Note. As in Art. 59, instead of taking the smallest positive angle, we might take α to be any one angle having for its cosine the given quantity k. The general value of θ satisfying $\cos\theta = \cos\alpha$ as obtained above, would not be affected at all.

61. General expression of all angles having the same tangent *(or cotangent).*

Let α be the smallest positive angle such that its tangent is equal to a given quantity k; and let θ be any other angle whose tangent is equal to k.

Then, $\tan\theta = \tan\alpha$,

or, $$\frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha} = 0.$$

or, $$\frac{\sin\theta\cos\alpha - \cos\theta\sin\alpha}{\cos\theta\cos\alpha} = 0,$$

or, $$\frac{\sin(\theta-\alpha)}{\cos\theta\cos\alpha} = 0.$$

$\therefore$ $\sin(\theta - \alpha) = 0$,

i.e., $\theta - \alpha =$ any multiple of $\pi = n\pi$.

$\therefore$ $\theta = \alpha + n\pi$. ... (1)

The factor $\dfrac{1}{\cos\theta\cos\alpha}$ cannot be zero, for cosine of an angle cannot have an infinitely large value.

It is also evident as in the previous case that all angles having the same cotangent as that of α are given by the expression (1).

Hence, *all angles having the same tangent or cotangent as that of α are given by*

$$n\pi+\alpha$$

n being zero, or any integer, positive or negative.

Note. The remark below Art. 60 is applicable here also.

62. Special cases.

From Art. 59, considering both cases when n is odd or even, it may be easily seen that

if $\sin\theta=1=\sin\dfrac{\pi}{2}$, $\quad\theta=2n\pi+\dfrac{\pi}{2}=(4n+1)\dfrac{\pi}{2}$

and if $\sin\theta=-1=\sin\left(-\dfrac{\pi}{2}\right)$, $\theta=2n\pi-\dfrac{\pi}{2}=(4n-1)\dfrac{\pi}{2}$

or, $\quad=(4k+3)\dfrac{\pi}{2}$

where n (or $k=n-1$) *is zero, or any integer, positive or negative.*

Similarly, from Art. 60, it may be seen that

if $\cos\theta=1$, $\theta=2n\pi$

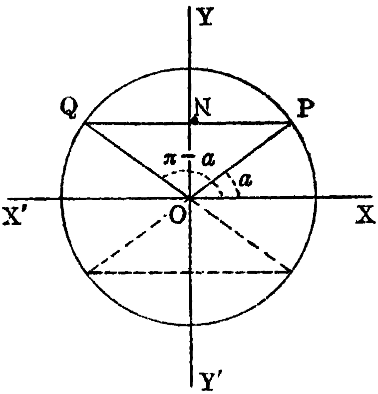
and if $\cos\theta=-1$, $\theta=(2n+1)\pi$

n being zero, or any integer, positive or negative.

These are the usual forms in which the above special cases are used in practice.

63. Geometrical Treatment.

(*i*) *Geometrical construction of an angle whose sine (or cosecant) is given, and to obtain a general expression of all such angles.*



Let the sine of an angle be given equal to 'a'.

Taking the perpendicular lines XOX' and YOY' for reference, draw a circle of unit radius with centre O.

Measure off $ON = a$ along OY (or along OY' if a be negative). Through N draw a straight line PNQ parallel to XOX' meeting the circle at P and Q.

Then, $\angle POX = a$ say, is one of the required angles, for

$$\sin a = \sin OPN = \frac{ON}{OP} = \frac{a}{1} = a.$$

Another angle with the same sine, as is apparent from the figure, is $\angle QOX = \pi - a$ (or $3\pi - a$ if $a = ON$ be negative, which is trigonometrically the same as $\pi - a$).

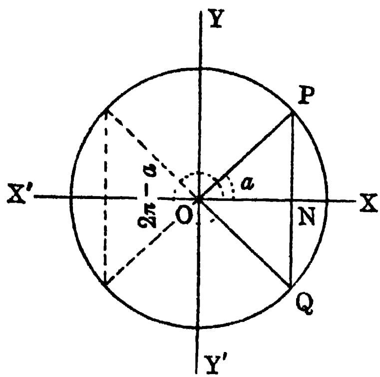
'a' being given in magnitude and sign, the position of N on YOY' is fixed and thus in one revolution, *i.e.*, from 0 to

2π there are, as is clear from the figure, only two angles a and $\pi - a$ having the given sine.*

Now, the addition or subtraction of any multiple of 2π makes no difference in the values of the trigonometrical ratios of an angle (See *Art. 28*).

Hence, all the angles having the same sine as that of a are contained in the formulæ $2m\pi + a$ and $2m\pi + \pi - a$ *i.e.*, $(2m + 1)\pi - a$, where m is zero, or any integer, positive or negative Both the sets of angles are evidently included in the formula $\mathbf{n\pi + (-1)^n}\ \boldsymbol{a}$, n being zero, or any integer, positive or negative.

(*ii*) *Angles with given cosine (or secant).*



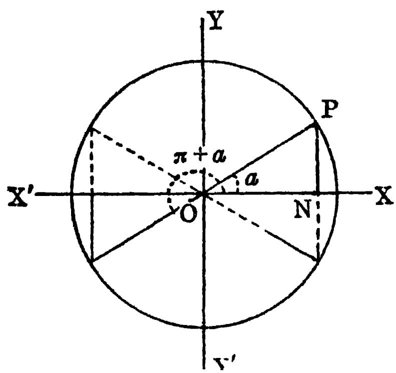
Let the given cosine be 'a'. As before, measure off $ON = a$ along OX (or along OX' if 'a' be negative), and through N draw PNQ parallel to YOY' to meet the circle with centre O and radius unity, at P and Q.

* In the same quadrant there cannot be two distinct angles (without being coterminals) having the same sine, for the corresponding triangles will then be congruent.

Let $\angle POX = a$. Then, a is a required angle. Also from the figure, the only angles in the first four quadrants which have the given cosine are a and $2\pi - a$.

Adding or subtracting multiples of of 2π to these, all the angles having the same cosine as that of a are given by $2m\pi + a$ or $2m\pi + 2\pi - a$, both of which are included in the formula $\mathbf{2n\pi \pm a}$, n being zero, or any integer, positive or negative.

(*iii*) *Angles with given tangent (or cotangent).*



Let 'a' be the given tangent. Along OX or OX' measure off ON of unit length, and then measure off NP perpendicular to it of length whose numerical value is 'a'. If 'a' be positive, both ON and NP will be positive, or both will be negative, and so the $\angle XOP$ will be either in the first or in the third quadrant. If 'a' be negative, the angle will be either in the second or in the fourth quadrant. In any case there are only two angles, within one revolution, *i.e.*, from 0 to 2π as is apparent from the figure, with the given tangent.*

* The ratio $PN : ON$ being given, and the included angle PNO being right, the triangle PNO constructed remains always similar to itself and so in the same quadrant $\angle PON$ of the triangle is unique.

One of the angles being α, the other is evidently (from the figure) $\pi+\alpha$. Adding or subtracting multiples of 2π, all the angles having the same tangent as that of α are given by $2m\pi+\alpha$ or $2m\pi+\pi+\alpha$ both of which are included in the formula $n\pi+\alpha$ where n is zero, or any integer, positive or negative, odd or even.

Ex. 1. *Solve* $2(\cos^2\theta-\sin^2\theta)=1$.

The given equation can be written as

$$2\cos 2\theta=1. \qquad \therefore \quad \cos 2\theta=\tfrac{1}{2}=\cos\tfrac{1}{3}\pi.$$

$$\therefore \quad 2\theta=2n\pi\pm\tfrac{1}{3}\pi. \qquad \therefore \quad \theta=n\pi\pm\tfrac{1}{6}\pi.$$

Note. It may be observed that a trigonometrical equation can be solved in several ways ; and the results though different in forms will give the same series of angles. To illustrate this we work out the above example in another way.

The equation can also be written in the form

$$2(\cos^2\theta-1+\cos^2\theta)=1, \text{ or, } 4\cos^2\theta=3.$$

$$\therefore \quad \cos\theta=\pm\frac{\sqrt{3}}{2}=\cos\frac{\pi}{6}, \text{ or, } \cos\frac{5\pi}{6},$$

$$\therefore \quad \theta=2m\pi\pm\frac{\pi}{6}, \text{ or, } 2m\pi\pm\frac{5\pi}{6}.$$

Now, $2m\pi\pm\frac{5\pi}{6}=(2m+1)\pi-\frac{\pi}{6}$, or $(2m-1)\pi+\frac{\pi}{6}$.

All the four sets of solutions, m being any integer, can be included in the expression $n\pi\pm\frac{1}{6}\pi$, in which form the result has already been obtained by the previous process.

Ex. 2. *Solve* $4\cos^2x+6\sin^2x=5$.

The equation can be written as

$$4\cos^2x+6\sin^2x=5(\sin^2x+\cos^2x).$$

$$\therefore \quad \sin^2x=\cos^2x, \text{ or, } \tan^2x=1.$$

$$\therefore \quad \tan x=\pm 1. \quad \therefore \quad x=n\pi\pm\tfrac{1}{4}\pi.$$

Note. Equations of the form $a\cos^2x+b\sin^2x=c$ can be easily solved by the above method, or by expressing sine in terms of cosine or cosine in terms of sine.

Ex. 3. *Solve* $2\sin^2x+\sin^2 2x=2$. [*C. U. 1940*]

The given equation can be written as

$$2(1-\sin^2x)-\sin^2 2x=0, \text{ or, } 2\cos^2x-4\sin^2x\cos^2x=0.$$

or, $\quad 2\cos^2 x\,(1-2\sin^2 x)=0$, or, $\cos^2 x \cos 2x = 0$.

$\therefore\quad$ either $\cos x = 0$, *i.e.*, $x = n\pi + \frac{1}{2}\pi$,

or, $\quad \cos 2x = 0$, *i.e.*, $2x = 2n\pi \pm \frac{1}{2}\pi$. $\quad\therefore\quad x = n\pi \pm \frac{1}{4}\pi$.

Ex. 4. *Solve* $\cos\theta - \sin\theta = \dfrac{1}{\sqrt{2}}$.

Dividing both sides of the equation by $\sqrt{1^2+1^2}$, *i.e.*, $\sqrt{2}$, we have

$$\cos\theta.\ \frac{1}{\sqrt{2}} - \sin\theta.\ \frac{1}{\sqrt{2}} = \frac{1}{2},$$

i.e., $\cos\theta\cos\frac{1}{4}\pi - \sin\theta\sin\frac{1}{4}\pi = \frac{1}{2}$.

$\therefore\quad \cos(\theta+\frac{1}{4}\pi) = \cos\frac{1}{3}\pi. \quad\therefore\quad \theta + \frac{1}{4}\pi = 2n\pi \pm \frac{1}{3}\pi.$

$\therefore\quad \theta = 2n\pi + \frac{1}{12}\pi, \quad$ or, $\quad 2n\pi - \frac{7}{12}\pi$.

Note. Extraneous solutions.

In general, as pointed out in Ex. 1 above, the same trigonometrical equation may be solved by different methods, and the forms of the result we arrive at, though apparently different in some cases, are ultimately equivalent. In some cases, however, we may be tempted to solve a trigonometrical equation by methods which have flaws in them, leading to solutions which include in addition to the correct solutions, some extraneous solutions which do not satisfy the given equation. The given equation which is of the type $a\cos\theta + b\sin\theta = c$ is an example. We proceed to demonstrate it as follows :

Here, $\quad \cos\theta - \dfrac{1}{\sqrt{2}} = \sin\theta$.

$\therefore\quad \cos^2\theta - \sqrt{2}\cos\theta + \frac{1}{2} = \sin^2\theta = 1 - \cos^2\theta$,

whence $2\cos^2\theta - \sqrt{2}\cos\theta - \frac{1}{2} = 0$.

$\therefore\quad \cos\theta = \dfrac{\sqrt{2}\pm\sqrt{2+4}}{4} = \dfrac{1\pm\sqrt{3}}{2\sqrt{2}} = \cos\dfrac{\pi}{12}$, or, $\cos\dfrac{7\pi}{12}$,

$\therefore\quad \theta = 2n\pi \pm \frac{1}{12}\pi$, or, $2n\pi \pm \frac{7}{12}\pi$.

But it can be easily seen on substitution that

$2n\pi - \frac{1}{12}\pi$ and $2n\pi + \frac{7}{12}\pi$ do not satisfy the given equation. *The error in the method lies in squaring the equation* as we have done; for the squared equation includes the equation $\cos\theta - \dfrac{1}{\sqrt{2}} = -\sin\theta$, *i.e.*, $\cos\theta + \sin\theta = \dfrac{1}{\sqrt{2}}$ of which the solutions are $2n\pi - \frac{1}{12}\pi$ and $2n\pi + \frac{7}{12}\pi$.

Equations of this type are therefore best solved as in the next example, and not by squaring.

Thus, *while solving any trigonometrical equation it is always advisable to verify the roots obtained*; for thereby extraneous roots, if any, can be easily detected.

Ex. 5. *Solve* $a \cos \theta + b \sin \theta = c$. $(c \not> \sqrt{a^2+b^2})$

Put $a = r \cos \alpha$, $b = r \sin \alpha$, choosing the smallest positive value of α, keeping r positive.

Then, $r = \sqrt{a^2+b^2}$ and $\sin \alpha = \dfrac{b}{\sqrt{a^2+b^2}}$

$$\text{and } \cos \alpha = \frac{a}{\sqrt{a^2+b^2}}.$$

The signs of a and b will determine the quadrant in which α lies, and a and b being given, r and α are definitely known.

The equation now becomes

$$r \cos (\theta - \alpha) = c,$$

$$\text{or, } \cos (\theta - \alpha) = \frac{c}{\sqrt{a^2+b^2}} = \cos \beta,$$

where β is the smallest positive angle whose cosine is $\dfrac{c}{\sqrt{a^2+b^2}}$, and a, b, c being known, β is also known.

Hence, $\theta - \alpha = 2n\pi \pm \beta$, or, $\theta = 2n\pi + \alpha \pm \beta$.

Note. An angle which is introduced in a trigonometrical work to facilitate calculations is called a *subsidiary angle*. Thus, α and β are here subsidiary angles.

Ex. 6. *Solve* $4 \cos x + 5 \sin x = 5$, *given* $\tan 51^\circ 21' = \frac{5}{4}$.

Dividing both sides of the given equation by $\sqrt{4^2+5^2}$, *i.e.*, by $\sqrt{41}$, we get

$$\frac{4}{\sqrt{41}} \cos x + \frac{5}{\sqrt{41}} \sin x = \frac{5}{\sqrt{41}}. \quad \cdots \quad (1)$$

Since, $\tan 51^\circ 21' = \frac{5}{4}$,

$$\therefore \quad \sin 51^\circ 21' = \frac{5}{\sqrt{41}}, \ \cos 51^\circ 21' = \frac{4}{\sqrt{41}}.$$

$\therefore$ (1) reduces to

$$\cos x \cos 51^\circ 21' + \sin x \sin 51^\circ 21' = \sin 51^\circ 21',$$

or, $\cos(x - 51^\circ 21') = \sin 51^\circ 21' = \cos 38^\circ 39$.

$\therefore$ $x - 51^\circ 21' = 2n\pi \pm 38^\circ 39'$.

$\therefore$ $x = 2n\pi + 90^\circ$, or, $2n\pi + 12^\circ 42'$.

Ex. 7. (i) *Solve*, $2\sin^2 x + \sin^2 2x = 2$ *for* $-\pi < x < \pi$.

From Ex. 3 above, we see that $x = n\pi + \frac{1}{2}\pi$ ... (1)

or, $x = n\pi \pm \frac{1}{4}\pi$. ... ... (2)

Putting $n = 0, -1$ in (1), we get $x = \frac{1}{2}\pi, -\frac{1}{2}\pi$, which lie in the given interval. Putting $n = 0, 1, -1$ in (2), we get $x = \pm\frac{1}{4}\pi, \frac{3}{4}\pi, -\frac{3}{4}\pi$ which also lie in the given interval.

Hence, the required values of x are $\pm\frac{1}{4}\pi, \pm\frac{1}{2}\pi, \pm\frac{3}{4}\pi$.

(ii) *Solve* $\cos\theta + \sqrt{3}\sin\theta = 2$

for $-2\pi < \theta < 2\pi$ *and* $3\pi < \theta < 5\pi$.

Dividing both sides of the equation by $\sqrt{1+3}$, *i.e.*, 2, we have

$$\cos\theta \cdot \frac{1}{2} + \sin\theta \cdot \frac{\sqrt{3}}{2} = 1,$$

i.e., $\cos\theta . \cos\frac{1}{3}\pi + \sin\theta . \sin\frac{1}{3}\pi = 1$,

i.e., $\cos(\theta - \frac{1}{3}\pi) = 1$.

$\therefore$ $\theta - \frac{1}{3}\pi = 2n\pi$, *i.e.*, $\theta = 2n\pi + \frac{1}{3}\pi$.

Putting $n = 0, -1$, we get $\theta = \frac{1}{3}\pi, -\frac{5}{3}\pi$ which lie in the 1st interval.

Again, putting $n = 1, 2$, we get $\theta = \frac{7}{3}\pi, \frac{13}{3}\pi$, which lie in the 2nd interval.

Ex. 8. *Solve* $\tan ax = \cot bx$.

Here, $\tan ax = \cot bx = \tan(\frac{1}{2}\pi - bx)$.

$\therefore$ $ax = n\pi + \frac{1}{2}\pi - bx$.

$$\therefore \quad x = \frac{2n+1}{a+b} \cdot \frac{\pi}{2}.$$

Examples XI

Solve the following equations (*Ex. 1 to 23*) :—

1. $\cot^2 x + \text{cosec}^2 x = 3$.

2. (i) $2\cos^2\theta + 4\sin^2\theta = 3$.

(ii) $\tan^2\theta = 3\,\text{cosec}^2\theta - 1$. [*C. U. 1939*]

3. $\tan x - \cot x = \text{cosec}\, x$.

4. $\cot x - \cot 2x = 2$.

5. $2\sin\theta\tan\theta + 1 = \tan\theta + 2\sin\theta$.

6. $\sin 5\theta + \sin\theta = \sin 3\theta$.

7. $\sin m\theta + \sin n\theta = 0$.

8. $\cos x + \cos 3x + \cos 5x + \cos 7x = 0$.

9. $\cot 2x = \cos x + \sin x$.

10. $\sin x + \cos x = \sqrt{2}$, for $-\pi < x < \pi$.

11. $\sin 2x\tan x + 1 = \sin 2x + \tan x$.

12. $\cot x - \tan x = 2$. [*C. U. 1934, '37*]

13. $\sin x + \sqrt{3}\cos x = \sqrt{2}$. [*C. U. 1938, '47*]

14. $2\sin x\sin 3x = 1$.

15. $\sin\theta + 2\cos\theta = 1$. [*C. U. 1933*]

16. $\tan x + \tan 2x + \tan 3x = \tan x\tan 2x\tan 3x$.

17. $\tan(\frac{1}{4}\pi + \theta) + \tan(\frac{1}{4}\pi - \theta) = 4$. [*C. U. 1949*]

18. $\tan x + \tan 2x + \tan x\tan 2x = 1$. [*C. U. 1941, '45*]

19. $\cos\theta + \sqrt{3}\sin\theta = \sqrt{2}$. [*C. U. 1944*]

20. $\sqrt{3}\cos x + \sin x = 1$, for $-2\pi < x < 2\pi$.

21. $\cos 2x = \cos x\sin x$.

22. $2\cot x + \sin x = 2\,\text{cosec}\, x$.

23. $\cos x + \sin x = \cos 2x + \sin 2x$. [*C. U. 1943*]

24. Solve $2\sin^2 x + \sin x = 3$; and find all the angles between 0° and 1000° which satisfy it.

25. Find the solution of the equations (general solution is not required)

$$\tan x + \tan y = 2$$
$$2\cos x\cos y = 1.$$

26. If $\tan ax - \tan bx = 0$, show that the values of x form a series in A. P.

27. Solve

(i) $\cos 3x + \cos 2x + \cos x = 0.$ [*C. U. 1941, '46*]

(ii) $\cos 9x \cos 7x = \cos 5x \cos 3x$, $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$.

(iii) $\tan x + \tan 2x + \tan 3x = 0.$ [*A. I. 1941*]

(iv) $\cos x - \sin x = \cos a + \sin a.$ [*B. H. U. 1938*]

(v) $\cos^3 x - \cos x \sin x - \sin^3 x = 1.$

(vi) $\cos 6x + \cos 4x = \sin 3x + \sin x.$

(vii) $\dfrac{\sin a}{\sin 2x} + \dfrac{\cos a}{\cos 2x} = 2.$

28. Solve $5 \cos \theta + 2 \sin \theta = 2$, given $\tan 68° \, 12' = 2\frac{1}{2}$.

29. Find those pairs of solutions of the following equations which correspond to positive solutions less than 2π of each individual equation :—

(i) $\sin (\alpha - \beta) = 0$; $\sin (\alpha + \beta) = 1.$

(ii) $\sin (\alpha - \beta) = \cos (\alpha + \beta) = \frac{1}{2}.$

30. If $\sin A = \sin B$, $\cos A = \cos B$, prove that either A and B are equal or they differ by some multiple of four right angles. [*C. U. 1935*]

31. Show that the three equations

$$\sin^2\theta = \sin^2 a,\ \cos^2\theta = \cos^2 a,\ \tan^2\theta = \tan^2 a$$

are all identical and the solution is always $n\pi \pm a$.

32. Show that the same two series of angles are given by the equations

$$x + \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{6} \text{ and } x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}.$$

CHAPTER XII

INVERSE CIRCULAR FUNCTIONS

64. The equation $\sin\theta = x$ means that θ is an angle whose sine is x. It is often convenient to express this statement *inversely* by writing $\theta = \sin^{-1}x$. Thus, the symbol $\sin^{-1}x$ denotes an angle whose sine is x. Hence, $\sin^{-1}x$ is an angle, whereas $\sin\theta$ is a number. The two relations $\sin\theta = x$ and $\theta = \sin^{-1}x$ are identical, if one is given the other follows. The symbol $\sin^{-1}x$ is usually read as "*sine inverse x*". Sometimes it is also denoted by *arc sin x*.

Note. $\sin^{-1}x$ must not be confused with $(\sin x)^{-1}$, *i.e.*, $\dfrac{1}{\sin x}$.

65. We know that if θ be any one angle whose sine is equal to x, then sines of all the angles given by $n\pi + (-1)^n\theta$ are equal to x. Hence, $\sin^{-1}x$ has got an infinite number of values, and as such, $\sin^{-1}x$ is a *multiple-valued function.*

Hence, the *general value of* $\sin^{-1}x = n\pi + (-1)^n \sin^{-1}x$ where on the right-hand side $\sin^{-1}x$ stands for any particular angle whose sine is x.

Similarly, the *general value of*

$$\cos^{-1}x = 2n\pi \pm \cos^{-1}x$$

and of $\tan^{-1}x = n\pi + \tan^{-1}x$.

The smallest numerical value, either positive or negative, of θ is called the *principal value* of $\sin^{-1}x$. Thus, the principal value of $\sin^{-1}\frac{1}{2}$ is $30°$. If corresponding to the same ratio, there are two numerically equal angles, one positive and the other negative, it is customary to take the positive angle as the principal value; thus, the principal value of $\cos^{-1}\frac{1}{2}$ is $60°$, and not $(-60°)$ although $\cos(-60°) = \frac{1}{2}$.

In all numerical examples, the principal value is generally taken.

$\cos^{-1}x$, $\tan^{-1}x$, $\operatorname{cosec}^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$ have similar significance and all properties as those of $\sin^{-1}x$. These expressions are called **Inverse Circular Functions.**

66. If $\sin\theta = x$, then $\theta = \sin^{-1}x$, *i.e.*, $\boldsymbol{\theta = \sin^{-1}\sin\theta}$.

Similarly, $\boldsymbol{\theta = \cos^{-1}\cos\theta = \tan^{-1}\tan\theta}$; etc.

Again, if $\theta = \sin^{-1}x$, $\sin\theta = x$, *i.e.*, $\mathbf{\sin\sin^{-1}x = x}$.

Similarly, $\mathbf{\cos\cos^{-1}x = x}$: $\mathbf{\tan\tan^{-1}x = x}$; etc.

Also, we have

$$\mathbf{cosec^{-1}x = sin^{-1}\frac{1}{x}\ ;\ cot^{-1}x = tan^{-1}\frac{1}{x}\ ;\ sec^{-1}x = cos^{-1}\frac{1}{x}}.$$

Let $\operatorname{cosec}^{-1}x = \theta$; then $\operatorname{cosec}\theta = x$.

$$\therefore \quad \sin\theta = \frac{1}{\operatorname{cosec}\theta} = \frac{1}{x}.$$

Hence, $\theta = \sin^{-1}\frac{1}{x}$, and therefore, $\operatorname{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$.

In the same way we have, $\operatorname{cosec}^{-1}\frac{1}{x} = \sin^{-1}x$.

The other relations follow similarly.

67. As all the trigonometrical ratios can be expressed in terms of any one, similarly all the inverse trigonometrical ratios can be expressed in terms of any one inverse ratio.

Thus, let $\sin^{-1}x = \theta$; then $\sin\theta = x$,

$$\therefore \quad \cos\theta = \sqrt{1-x^2}\ ;\ \tan\theta = \frac{x}{\sqrt{1-x^2}}\ ;\ \cot\theta = \frac{\sqrt{1-x^2}}{x}\ ;$$

$$\sec\theta = \frac{1}{\sqrt{1-x^2}} \text{ and } \operatorname{cosec}\theta = \frac{1}{x}.$$

$$\therefore \quad \theta = \sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$$

$$= \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \operatorname{cosec}^{-1}\frac{1}{x}$$

68. *To prove that*

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$.

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$.

(iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$.

(i) Let $\sin^{-1}x = \theta$; then $\sin\theta = x$.

Now, $\sin\theta = \cos(\frac{1}{2}\pi - \theta)$.

$\therefore$ $\cos(\frac{1}{2}\pi - \theta) = x$ and hence $\cos^{-1}x = \frac{1}{2}\pi - \theta$.

Therefore, $\sin^{-1}x + \cos^{-1}x = \theta + \frac{1}{2}\pi - \theta = \frac{1}{2}\pi$.

(ii) Let $\tan^{-1}x = \theta$; then $\tan\theta = x$.

Now, $\tan\theta = \cot(\frac{1}{2}\pi - \theta)$.

$\therefore$ $\cot(\frac{1}{2}\pi - \theta) = x$. $\therefore$ $\cot^{-1}x = \frac{1}{2}\pi - \theta$.

$\therefore$ $\tan^{-1}x + \cot^{-1}x = \theta + \frac{1}{2}\pi - \theta = \frac{1}{2}\pi$.

(iii) Let $\operatorname{cosec}^{-1}x = \theta$; then $\operatorname{cosec}\theta = x$.

Now, $\operatorname{cosec}\theta = \sec(\frac{1}{2}\pi - \theta)$.

$\therefore$ $\sec(\frac{1}{2}\pi - \theta) = x$. $\therefore$ $\sec^{-1}x = \frac{1}{2}\pi - \theta$.

$\therefore$ $\operatorname{cosec}^{-1}x + \sec^{-1}x = \theta + \frac{1}{2}\pi - \theta = \frac{1}{2}\pi$.

69. *To prove that*

(i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$.

(ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$.

Let $\tan^{-1}x = \alpha$; and $\tan^{-1}y = \beta$;

then $\tan\alpha = x$; and $\tan\beta = y$.

Now, $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{x+y}{1-xy}$.

$\therefore \quad \alpha+\beta=\tan^{-1}\dfrac{x+y}{1-xy},$

i.e., $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\dfrac{x+y}{1-xy}.$

Again, $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\dfrac{x-y}{1+xy}.$

$\therefore \quad \alpha-\beta=\tan^{-1}\dfrac{x-y}{1+xy},$

i.e., $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\dfrac{x-y}{1+xy}.$

Note. It can be easily proved as above that

$$\cot^{-1}x\pm\cot^{-1}y=\cot^{-1}\frac{xy\mp 1}{y\pm x}.$$

70. *To prove that*

$$\mathbf{\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\frac{x+y+z-xyz}{1-yz-zx-xy}}.$$

Let $\tan^{-1}x=\alpha$; $\tan^{-1}y=\beta$; $\tan^{-1}z=\gamma$.

$\therefore \quad \tan\alpha=x, \quad \tan\beta=y, \quad \tan\gamma=z.$

Now, $\tan(\alpha+\beta+\gamma)$

$$=\frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\beta\tan\gamma-\tan\gamma\tan\alpha-\tan\alpha\tan\beta}$$

$$=\frac{x+y+z-xyz}{1-yz-zx-xy}.$$

Hence, $\quad \alpha+\beta+\gamma=\tan^{-1}\dfrac{x+y+z-xyz}{1-yz-zx-xy}.$

Since, $\alpha+\beta+\gamma=\tan^{-1}x+\tan^{-1}y+\tan^{-1}z$, the required result follows.

Note. This relation can also be deduced by applying twice the formula of Art. 69. Thus,

Left side $=(\tan^{-1}x+\tan^{-1}y)+\tan^{-1}z$

$=\tan^{-1}\dfrac{x+y}{1-xy}+\tan^{-1}z$; **now again apply Art. 69.**

71. In fact for most of the formulæ involving ordinary circular functions, corresponding relations connecting the inverse circular functions can be easily deduced. In addition to those given above, some are illustrated in the following examples.

Ex. 1. *Show that*

(i) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\}$.

(ii) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\{xy \mp \sqrt{(1-x^2)(1-y^2)}\}$.

(i) Let $\sin^{-1}x = \alpha$. $\therefore$ $\sin\alpha = x$ and $\cos\alpha = \sqrt{1-x^2}$;

also let $\sin^{-1}y = \beta$. $\therefore$ $\sin\beta = y$ and $\cos\beta = \sqrt{1-y^2}$.

Now, $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$$= x\sqrt{1-y^2} \pm y\sqrt{1-x^2}.$$

$\therefore$ $\alpha \pm \beta = \sin^{-1}\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\}$.

Since, $\alpha \pm \beta = \sin^{-1}x \pm \sin^{-1}y$, the required result follows.

(ii) These relations follow similarly from the value of $\cos(\alpha \pm \beta)$.

Ex. 2. *Show that*

(i) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$.

(ii) $2\cos^{-1}x = \cos^{-1}(2x^2-1)$.

(iii) $2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2}$.

(i) Let $\sin^{-1}x = \alpha$. $\therefore$ $\sin\alpha = x$, $\cos\alpha = \sqrt{1-x^2}$.

Now, $\sin 2\alpha = 2\sin\alpha\cos\alpha = 2x\sqrt{1-x^2}$.

$\therefore$ $2\alpha = \sin^{-1}(2x\sqrt{1-x^2})$.

Since, $\alpha = \sin^{-1}x$, the requird result follows.

(ii) & (iii). These relations follow similarly from the corresponding values of $\cos 2\alpha$ in terms of $\cos\alpha$ and $\tan 2\alpha$ in terms of $\tan\alpha$. [See *Art. 43*]

Note. The above three relations can also be deduced by putting x for y in the values of $\sin^{-1}x+\sin^{-1}y$, $\cos^{-1}x+\cos^{-1}y$ and $\tan^{-1}x+\tan^{-1}y$.

Ex. 3. *Show that*

(i) $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$.

(ii) $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$.

(iii) $3\tan^{-1}x=\tan^{-1}\dfrac{3x-x^3}{1-3x^2}$. [*C. U. 1938*]

(i) Let $\sin^{-1}x=\theta$; then $\sin\theta=x$.

Now, $\sin 3\theta=3\sin\theta-4\sin^3\theta=3x-4x^3$.

$\therefore$ 3θ, *i.e.*, $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$.

(ii) & (iii). These relations follow similarly from the corresponding values of $\cos 3\theta$ in terms of $\cos\theta$ and of $\tan 3\theta$ in terms of $\tan\theta$. [See *Art. 44*]

Note. The result (iii) may also be deduced by putting $y=z=x$ in the formula of Art. 70.

Ex. 4. *Show that*

$$2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}=\cos^{-1}\frac{1-x^2}{1+x^2}=\tan^{-1}\frac{2x}{1-x^2}.$$

Let $\tan^{-1}x=\theta$, $\therefore$ $\tan\theta=x$.

Since, $\sin 2\theta=\dfrac{2\tan\theta}{1+\tan^2\theta}=\dfrac{2x}{1+x^2}$, [*Art. 45, Ex. 1*]

$\therefore$ 2θ, *i.e.*, $2\tan^{-1}x=\sin^{-1}\dfrac{2x}{1+x^2}$.

Since, $\cos 2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}=\dfrac{1-x^2}{1+x^2}$,

and $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}=\dfrac{2x}{1-x^2}$,

the remaining relations follow similarly.

Ex. 5. *Show that*

$$\tan^{-1}\frac{a-b}{1+ab}+\tan^{-1}\frac{b-c}{1+bc}+\tan^{-1}\frac{c-a}{1+ca}=0.$$

1st term of left side $=\tan^{-1}a-\tan^{-1}b$ [*By Art. 69 (ii)*]

2nd ... ... $=\tan^{-1}b-\tan^{-1}c.$

3rd ... ... $=\tan^{-1}c-\tan^{-1}a$

Hence, adding up the three terms, the required result follows.

Ex. 6. *Show that*

$$2\tan^{-1}\tfrac{1}{5}+\tan^{-1}\tfrac{1}{4}=\tan^{-1}\tfrac{32}{43}.$$

Since, $2\tan^{-1}x=\tan^{-1}\dfrac{2x}{1-x^2}$, [See *Ex. 4*]

$$2\tan^{-1}\tfrac{1}{5}=\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{5^2}}=\tan^{-1}\tfrac{5}{12}.$$

$\therefore$ Left side $=\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{1}{4}=\tan^{-1}\dfrac{\frac{5}{12}+\frac{1}{4}}{1-\frac{5}{48}}=\tan^{-1}\frac{32}{43}.$

Ex. 7. *Solve*

$$\sin^{-1}\frac{2a}{1+a^2}+\sin^{-1}\frac{2b}{1+b^2}=2\tan^{-1}x.$$

[*C. U. 1947*]

Since, $\sin^{-1}\dfrac{2x}{1+x^2}=2\tan^{-1}x$, [See *Ex. 4*]

$\therefore$ Left side $=2\tan^{-1}a+2\tan^{-1}b.$

$\therefore$ the equation reduces to

$$2\tan^{-1}x=2\tan^{-1}a+2\tan^{-1}b.$$

$\therefore\quad \tan^{-1}x=\tan^{-1}a+\tan^{-1}b=\tan^{-1}\dfrac{a+b}{1-ab}.$

$\therefore\quad x=\dfrac{a+b}{1-ab}.$

Ex. 8. *Solve*

$$\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}.$$

$$\text{Left side}=\tan^{-1}\frac{\dfrac{x-1}{x-2}+\dfrac{x+1}{x+2}}{1-\dfrac{x^2-1}{x^2-4}}=\tan^{-1}\frac{2x^2-4}{-3}.$$

$\therefore$ the equation reduces to

$$\tan^{-1}\frac{2x^2-4}{-3}=\frac{\pi}{4}=\tan^{-1}1$$

$$\therefore\ \frac{2x^2-4}{-3}=1 \text{ or, } 2x^2=1 \text{ or, } x=\pm\frac{1}{\sqrt{2}}.$$

Examples XII

Prove *Ex.* (*1 to 17*) that :—

1. (i) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{1}{4}\pi.$

(ii) $\tan^{-1}x+\tan^{-1}\dfrac{2x}{1-x^2}=\tan^{-1}\dfrac{3x-x^3}{1-3x^2}.$

(iii) $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{18}=\cot^{-1}3.$

2. $\tan^{-1}\frac{2}{11}+\cot^{-1}\frac{24}{7}=\tan^{-1}\frac{1}{2}.$

3. $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$

$=2(\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}).$

4. (i) $\tan^{-1}x+\cot^{-1}(x+1)=\tan^{-1}(x^2+x+1).$

(ii) $\tan^{-1}\dfrac{1}{p+q}+\tan^{-1}\dfrac{q}{p^2+pq+1}=\tan^{-1}\dfrac{1}{p}.$

5. $\tan^{-1}a-\tan^{-1}c=\tan^{-1}\dfrac{a-b}{1+ab}+\tan^{-1}\dfrac{b-c}{1+bc}.$

6. $\tan^{-1}\frac{3}{5}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{27}{11}.$

7. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\frac{1}{4}\pi.$

[*C. U. 1942*]

8. $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{1}{4}\pi.$ [*C. U. 1937*]

9. (i) $\sin(2\sin^{-1}x) = 2x\sqrt{1-x^2}$.

(ii) $\{\cos(\sin^{-1}x)\}^2 = \{\sin(\cos^{-1}x)\}^2$.

10. $\cos^{-1}x = 2\sin^{-1}\sqrt{\dfrac{1-x}{2}} = 2\cos^{-1}\sqrt{\dfrac{1+x}{2}}$.

11. $\tan^{-1}\sqrt{x} = \dfrac{1}{2}\cos^{-1}\dfrac{1-x}{1+x}$. [*C. U. 1943*]

12. $\sin^{-1}\sqrt{\dfrac{x-b}{a-b}} = \cos^{-1}\sqrt{\dfrac{a-x}{a-b}} = \tan^{-1}\sqrt{\dfrac{x-b}{a-x}}$.

13. $\tan^{-1}\dfrac{a-b}{1+ab} + \tan^{-1}\dfrac{b-c}{1+bc} + \tan^{-1}\dfrac{c-a}{1+ca}$

$$= \tan^{-1}\frac{a^2-b^2}{1+a^2b^2} + \tan^{-1}\frac{b^2-c^2}{1+b^2c^2} + \tan^{-1}\frac{c^2-a^2}{1+c^2a^2}.$$

14. $\sec^2(\tan^{-1}2) + \operatorname{cosec}^2(\cot^{-1}3) = 15$.

15. $\cot^{-1}(\tan 2x) + \cot^{-1}(-\tan 3x) = x$.

16. $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{1}{2}\pi$. [*C. U. 1941*]

17. $4(\cot^{-1}3 + \operatorname{cosec}^{-1}\sqrt{5}) = \pi$. [*C. U. 1939*]

18. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, show that

$$x + y + z = xyz.$$

19. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{1}{2}\pi$, show that

$$yz + zx + xy = 1.$$

20. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, show that

$$x^2 + y^2 + z^2 + 2xyz = 1.$$

21. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, show that

$$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz.$$

22. Find the values of

(i) $\sin(\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{3})$. [*C. U. 1935*]

(ii) $\cot(\tan^{-1}a + \cot^{-1}a)$.

(iii) $\tan\left(\frac{1}{2}\sin^{-1}\dfrac{2x}{1+x^2} + \frac{1}{2}\cos^{-1}\dfrac{1-y^2}{1+y^2}\right)$.

23. If $\tan^{-1} y = 4 \tan^{-1} x$, find y as an algebraic function of x.

24. If $\tan^{-1}x$, $\tan^{-1}y$, $\tan^{-1}z$ are in A.P., find out the algebraic relation between x, y, z. If in addition, x, y, z are also in A.P., prove that $x = y = z$. [$y \neq 0$, 1 or -1]

25. Solve the following equations :

(i) $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$.

(ii) $\tan^{-1}\dfrac{2x}{1-x^2} = \sin^{-1}\dfrac{2a}{1+a^2} - \cos^{-1}\dfrac{1-b^2}{1+b^2}$.

(iii) $\tan(\cos^{-1}x) = \sin(\tan^{-1} 2)$.

(iv) $\tan^{-1}\dfrac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x$.

(v) $\tan^{-1}\dfrac{x-1}{x+1} + \tan^{-1}\dfrac{2x-1}{2x+1} = \tan^{-1}\dfrac{23}{36}$.

(vi) $\sin^{-1}x + \sin^{-1} 2x = \dfrac{\pi}{3}$.

(vii) $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$.

(viii) $\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1} 3x$.

(ix) $\tan^{-1}\dfrac{2x}{1-x^2} + \cot^{-1}\dfrac{1-x^2}{2x} = \dfrac{\pi}{3}$.

(x) $\cot^{-1}(x-1) + \cot^{-1}(x-2) + \cot^{-1}(x-3) = 0$.

26. Show that

(i) $\cot^{-1}\dfrac{xy+1}{x-y} + \cot^{-1}\dfrac{yz+1}{y-z} + \cot^{-1}\dfrac{zx+1}{z-x} = 0$.

(ii) $\tan(\tan^{-1}x + \tan^{-1}y + \tan^{-1}z)$

$$= \cot(\cot^{-1}x + \cot^{-1}y + \cot^{-1}z).$$

(iii) $\tan^{-1}(\cot x) + \cot^{-1}(\tan x) = \pi - 2x$.

Miscellaneous Examples I

1. If $3 \sin \theta + 4 \cos \theta = 5$, show that $\tan \theta = \frac{3}{4}$.

2. If $a^2 \sec^2 x - b^2 \tan^2 x = c^2$, find cosec x.

3. If $x = r \cos \theta \cos \phi$, $y = r \cos \theta \sin \phi$, $z = r \sin \theta$, show that $$x^2 + y^2 + z^2 = r^2.$$

4. If $\sin \theta = \dfrac{x - y}{x + y}$, show that $\tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{y}{x}}$.

5. If $x = r \sin(\theta + 45°)$ and $y = r \sin(\theta - 45°)$, then $$x^2 + y^2 = r^2.$$

6. If $\cos(\alpha + \beta) \sin(\gamma + \theta) = \cos(\alpha - \beta) \sin(\gamma - \theta)$, then $$\tan \theta = \tan \alpha \tan \beta \tan \gamma.$$

Show that (*Ex. 7 to 9*) :—

7. $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4 \sin^2 \dfrac{x - y}{2}$.

8. $\sin A + \sin B + \sin C - \sin(A + B + C)$
$$= 4 \sin \frac{A + B}{2} \sin \frac{B + C}{2} \sin \frac{C + A}{2}.$$

9. $4 \sin \dfrac{A + B + C}{2} \sin \dfrac{B + C - A}{2} \sin \dfrac{C + A - B}{2} \sin \dfrac{A + B - C}{2}$
$$= 1 - \cos^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C.$$

10. If $\tan \beta = \dfrac{2 \sin \alpha \sin \gamma}{\sin(\alpha + \gamma)}$, then $\tan \alpha$, $\tan \beta$, $\tan \gamma$ are in harmonical progression.

11. If $\alpha + \beta + \gamma = (2n + 1)\dfrac{\pi}{2}$, then

(i) $\tan \beta \tan \gamma + \tan \gamma \tan \alpha + \tan \alpha \tan \beta = 1$.

(ii) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = \pm 4 \cos \alpha \cos \beta \cos \gamma$.

12. If the angles A, B, C be in A. P., then
$$\frac{\sin A - \sin C}{\cos C - \cos A} = \frac{\cos B}{\sin B}.$$

13. If $\operatorname{cosec} 2A + \operatorname{cosec} 2B + \operatorname{cosec} 2C = 0$, show that

$$\tan A + \tan B + \tan C + \cot A + \cot B + \cot C = 0.$$

14. If $\tan \alpha = \dfrac{a \sin \beta}{1 - a \cos \beta}$ and $\tan \beta = \dfrac{b \sin \alpha}{1 - b \cos \alpha}$,

then $$\frac{\sin \alpha}{\sin \beta} = \frac{a}{b}.$$

15. Show that

$$\tan \theta + 2 \tan 2\theta + 4 \tan 4\theta + 8 \cot 8\theta = \cot \theta.$$

16. If $\cos (\theta - \psi) \cos \phi = \cos (\theta - \phi + \psi)$, then $\tan \theta$, $\tan \phi$, $\tan \psi$ are in harmonical progression.

17. If $1 + \cos (y - z) + \cos (z - x) + \cos (x - y) = 0$, show that either $(y - z)$, or $(z - x)$, or $(x - y)$ is an odd multiple of π.

18. If $\sin \theta + \sin \phi = \sqrt{3} (\cos \phi - \cos \theta)$, show that

$$\sin 3\theta + \sin 3\phi = 0.$$

19. Eliminate α and β from

$$\sin \alpha + \sin \beta = a, \ \cos \alpha + \cos \beta = b, \ \cos (\alpha - \beta) = c.$$

20. If $A + B + C = \pi$, prove that

(i) $\tan B \tan C + \tan C \tan A + \tan A \tan B$
$$= 1 + \sec A \sec B \sec C.$$

(ii) $\cot A + \cot B + \cot C = \cot A \cot B \cot C$
$$+ \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C.$$

21. If $A + B + C = \pi$, and if

$$\sin^2 A + \sin^2 B + \sin^2 C = \sin B \sin C + \sin C \sin A + \sin A \sin B,$$ then $A = B = C$.

22. If A, B, C be the angles of a triangle, and if $\cot A + \cot B + \cot C = \sqrt{3}$, show that the triangle is equilateral.

23. If $\sec ax + \sec bx = 0$, show that the values of x form two series in A. P.

CHAPTER XIII

LOGARITHMS

72. Definition of Logarithm.

Logarithm of a number with respect to a given base is the index of the power to which the base is to be raised in order to give the number.

Mathematically if $a^x = N$, then 'x' is the index of the power to which 'a' (which is called the base) is raised to give 'N'. Hence, by definition, 'x' is the logarithm of 'N' with respect to the base 'a' and it is usually written as $x = log_a N$.

As a numerical example, $\log_2 8 = 3$, for $2^3 = 8$ *i.e.*, 3 is the power to which 2 is to be raised to give 8. Again, since $3^4 = 81$, $4 = \log_3 81$.

Any result involving indices can be expressed as a result in logarithm, and *vice versa.*

For example,

$$\text{if } p^q = r, \quad \text{then, } q = \log_p r,$$
$$\text{if } m^n = z^k, \text{ then } n = \log_m (z^k),$$
$$\text{or, } k = \log_z (m^n).$$

Similarly, if $\log_y x = z$,

$$\text{then} \qquad y^z = x.$$

It should be noted that the logarithms of the same number with respect to different bases will be different ; for example, to get the same number 64, we must raise 2 to the power 6, whereas we are to raise 4 to the power 3 and 8 to the power 2 only ; hence $\log_2 64 = 6$, $\log_4 64 = 3$, $\log_8 64 = 2$.

Thus, so long as the base is not stated, logarithm of a number has no meaning.

73. Special Cases.

We know from Algebra that if a be any real finite quantity, other than zero, then $a^0 = 1$.

Hence, $\log_a 1 = 0$; in other words,

(i) *logarithm of 1 with respect to any finite quantity (other than zero) as base, is zero.*

Again, a being any quantity, $a^1 = a$.

Hence, $1 = \log_a a$. In other words,

(ii) *logarithm of any number with respect to itself as base is unity.*

Note 1. If $a^x = 0$, then $x = -\infty$ if $a > 1$, and $x = +\infty$ if $a < 1$.

Thus, we have $\log_a 0 = \mp\infty$ according as $a >$ or < 1. Hence, *logarithm of zero to a base greater than unity is minus infinity, and to a base less than unity is plus infinity.*

Note 2. Since the equation $a^x = -n$ (a and n being real positive quantities), cannot be satisfied by any real value of x, whether positive or negative, provided we consider the principal value* only of a^x, therefore, *logarithm of a negative quantity* (in a system of logarithms whose base is a real positive quantity) *must be imaginary.*

74. Fundamental formulæ in logarithms.

From the definition it is clear that logarithms are but indices in another form. Hence, corresponding to the three fundamental results in the theory of indices in Algebra, namely that if a, x, y be any real quantities,

(i) $a^x \times a^y = a^{x+y}$,

(ii) $a^x \div a^y = a^{x-y}$ and

(iii) $(a^x)^y = a^{xy}$,

we get three fundamental laws of logarithms which are given below.

---

* See a treatise on Higher Trigonometry.

(i) $\log_a (m \times n) = \log_a m + \log_a n$

in other words, *logarithm of the product of two quantities is equal to the sum of their logarithms taken separately, base remaining the same always.*

Proof. Put $\log_a m = x$, $\log_a n = y$

and $\log_a (m \times n) = z$

then from definition,

$$a^x = m, \ a^y = n \text{ and } a^z = m \times n = a^x \times a^y = a^{x+y},$$

so that, $z = x + y$.

Replacing values,

$$\log_a (mn) = \log_a m + \log_a n.$$

Cor. $\log_a (m.n.p\ldots\ldots) = \log_a m + \log_a n + \log_a p + \cdots\cdots$

(ii) $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

in other words, *logarithm of the quotient of two numbers is equal to the difference of their logarithms* (logarithm of the numerator minus logarithm of the denominator).

Proof. Put $\log_a m = x$, $\log_a n = y$

and $\log_a \left(\frac{m}{n}\right) = z$.

Then, from definition,

$$a^x = m, \ a^y = n$$

$$\text{and } a^z = \frac{m}{n} = \frac{a^x}{a^y} = a^{x-y},$$

so that

$$z = x - y,$$

or replacing values,

$$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n.$$

(iii) $\log_a (m)^n = n \log_a m$.

Or, *logarithm of a power of a number is the product of the power and the logarithm of the number.*

Proof. Put $\log_a m = x$, and $\log_a (m)^n = z$.

Then, by definition,

$$a^x = m \text{ and}$$
$$a^z = (m)^n = a^{nx}.$$
$$\therefore \quad z = nx,$$

or replacing values,

$$\log_a (m)^n = n \log_a m.$$

Ex. 1. *Reduce to a simple form* $\log_a \dfrac{x^p y^q}{z^s}$.

$$\log_a \frac{x^p y^q}{z^s} = \log_a (x^p y^q) - \log_a (z^s)$$
$$= \log_a x^p + \log_a y^q - \log_a z^s$$
$$= p \log_a x + q \log_a y - s \log_a z.$$

Ex. 2. *Simplify* $\log_{10} \sqrt[3]{\frac{25}{88}}$.

$$\log_{10} \sqrt[3]{\frac{25}{88}} = \log_{10} \left(\frac{5^2}{8.11}\right)^{\frac{1}{3}} = \tfrac{1}{3} \log_{10} \frac{5^2}{2^3.11}$$
$$= \tfrac{1}{3} \log_{10} \frac{10^2}{2^5.11}$$
$$= \tfrac{1}{3} [\, \log_{10} 10^2 - \log_{10} (2^5.11) \,]$$
$$= \tfrac{1}{3} [\, 2 \log_{10} 10 - (\log_{10} 2^5 + \log_{10} 11) \,]$$
$$= \tfrac{1}{3} [\, 2 - 5 \log_{10} 2 - \log_{10} 11 \,].$$

75. Change of base.

There is a fourth standard formula whereby logarithms of numbers with respect to one base being given, those with respect to a different base may be obtained. The formula is

$$\mathbf{\log_a m = \log_b m \times \log_a b.}$$

Proof. Put $\log_a m = x$, $\log_b m = y$ and $\log_a b = z$.

Then, from definition,

$$a^x = m,\ b^y = m,\ a^z = b.$$

Hence, $a^x = m = b^y = (a^z)^y = a^{yz}$,

or, $x = yz$.

Replacing values,

$$\log_a m = \log_b m \times \log_a b.$$

Cor. 1. In the above result, put $m = a$. Then remembering that $\log_a a = 1$, we get

$$\mathbf{\log_b a \times \log_a b = 1.}$$

Since, the above relation is very important, we add here an *independent proof* of it.

Let $\log_b a = x$, and $\log_a b = y$.

Then, $b^x = a$ and $a^y = b$.

$\therefore\ a = b^x = (a^y)^x = a^{xy}$. $\therefore\ xy = 1$,

i.e., $\log_b a \times \log_a b = 1$,

or, $\mathbf{\log_b a = \dfrac{1}{\log_a b}}$.

Cor. 2. The result of the above article may be written with the help of Cor. 1, in the form

$$\mathbf{\log_a m = \log_b m / \log_b a.}$$

Thus, if logarithms of both m and a with respect to b be known, logarithm of m with respect to a is obtained.

76. Common system of logarithms.

For all practical purposes wherever logarithms are used for numerical calculations, the base is usually taken as 10. Logarithms of numbers with respect to the base 10 are

referred to as the *Common system* of logarithms. The advantage of the common system of logarithms for practical applications will be clear presently, from the Article 77, Theorems I & II.

Note. In higher mathematics, for *theoretical* investigations, another quantity 'e' (defined in books of Algebra), whose value is nearly 2·718..., is used as the base of logarithms, and logarithms to this base e are called *Napierian* logarithms.

With the help of the logarithmic series established in books on Algebra, Napierian logarithms of numbers are tabulated. The factor $\frac{1}{\log_e 10}$ which is known as the *modulus of the common system*, applied to the Napierian logarithms will convert them to common logarithms (See *Art. 75*). Thus, a table of common logarithms is prepared.

Henceforth, we shall proceed with the consideration of the common system of logarithms, and the base being understood to be 10, will not be written.

77. Characteristic and Mantissa *of common logarithms.*

It is only in very few cases that the logarithm of a number is integral. In most cases, however, the logarithm of a number is partly integral and partly fractional (or decimal).

Def. The integral portion of the logarithm of a number is called the *characteristic*, and the decimal portion is called the *mantissa*.

In case the logarithm of a number is negative, and partly integral and partly decimal, the decimal portion, *i.e.*, the mantissa is always kept positive by altering the integral part, *i.e.*, the characteristic suitably. Thus, the *mantissa part of the logarithm of a number is always positive.* For instance, if the logarithm of a number is $-2{\cdot}3$, we write it as $-3+{\cdot}7$ and call -3 as the characteristic and ·7 (and not $-{\cdot}3$) as the mantissa. $-3+{\cdot}7$ is often abbreviated in the form $\bar{3}{\cdot}7$.

Theorem I. *The* **characteristic** *of the common logarithm of (i) any number greater than 1 is positive, and numerically one less than the number of digits in the integral part of the quantity whose logarithm is sought; and (ii) of any positive* number less than 1, is negative, and numerically one greater than the number of zeroes immediately after the decimal point in the quantity whose logarithm is wanted.*

(i) Let the number be greater than unity.

Any number, say 7·209, which consists of 1 digit only in its integral part, lies between 1 and 10.

Now, $10^0 = 1$ and $10^1 = 10$.

Hence, if $10^x = 7·209$, clearly x must be greater than 0 and less than 1.

Thus, log 7·209 must be between 0 and 1, *i.e.*, of the form 0·..., having its characteristic 0.

Similarly, numbers of the type 53·0528, which consists of 2 digits in their integral parts must lie between 10 and 100 *i.e.*, between 10^1 and 10^2.

Hence, the index to which 10 should be raised to give 53·0528 must be greater than 1 and less than 2, *i.e.*, log 53·0528 must be of the form 1·... having the characteristic 1.

log 10 is 1, and 10 also falls in this catagory of two digits.

In the same way, a number which has n digits in its integral part lies between 10^{n-1} (which also has n *digits*) and 10^n (which has $n+1$ digits). Thus, the logarithms of such numbers must lie between $n-1$ and n, *i.e.*, $(n-1)$ + some positive proper fraction. Hence, the characteristic in such cases is $n-1$.

Hence, the result.

*Logarithms of negative numbers are easily seen to be imaginary, for there is no real power, positive or negative, to which 10 may be raised to give a negative result. [See *Note 2, Art. 73*]

(ii) Let the number be positive, and less than 1 (*i.e.*, between 0 and 1).

We notice that

$$10^{0} \qquad\qquad = 1$$

$$10^{-1} = \frac{1}{10} \qquad = \cdot 1$$

$$10^{-2} = \frac{1}{100} \qquad = \cdot 01$$

$$10^{-3} = \frac{1}{1000} \qquad = \cdot 001$$

$$10^{-4} = \frac{1}{10000} \qquad = \cdot 0001$$

etc. etc. etc.

Now, a number less than 1, with no zero immediately after the decimal point, like ·3015, must be greatar than ·1 and less than 1 ; hence, the power to which 10 must be raised to give such a number must lie between -1 and 0, *i.e.*, $= -1 +$ a positive proper fraction. Hence, such numbers have the characteristic of their logarithms $= -1$.

A decimal number with one zero immediately after the decimal point, like ·078005, lies between ·01 and ·1 which are respectively equal to 10^{-2} and 10^{-1}.

Hence, if $10^{x} = \cdot 078005$, x must lie between -1 and -2, *i.e.*, x is of the form $-1\cdot\ldots\ldots$ Writing the decimal part of x positively, in the form $-2+\cdot\ldots\ldots$, we notice that the integral part of x, *i.e.*, the characteristic of the logarithm of ·078005 is -2.

Similarly, the logarithms of numbers between ·01 and ·001 (*i.e.*, 10^{-2} and 10^{-3}) which must have two zeroes after the decimal point, lie between -2 and -3, *i.e.*, are of the form $-2\cdot\ldots\ldots = -3 + \cdot\ldots\ldots$, and so the characteristic in such cases is -3,

and so on.

Hence the result.

Theorem II. *All numbers, formed of the same digits in the same order, differing only in the positions of their decimal points, have the* **mantissæ** *of their logarithms same.*

This will be clear from an example. Let us take the numbers 835107, 835107000, 83·5107, ·835107, ·000835107 and 8351·07.

$$
\begin{aligned}
\text{Now,}\quad \log 835107000 &= \log (835107 \times 1000)\\
&= \log 835107 + \log 1000\\
&= \log 835107 + 3.\\
\text{Again,}\quad \log 83{\cdot}5107 &= \log \frac{835107}{10000}\\
&= \log 835107 - \log 10000\\
&= \log 835107 - 4.\\
\log {\cdot}835107 &= \log \frac{835107}{1000000} = \log 835107 - 6.\\
\log {\cdot}000835107 &= \log \frac{835107}{10^9} = \log 835107 - 9.\\
\log 8351{\cdot}07 &= \log \frac{835107}{100} = \log 835107 - 2.
\end{aligned}
$$

Thus, the logarithms of all the numbers here differ from the logarithm of 835107 by a whole number in each case and so must have their decimal parts, *i.e.*, their mantissæ the same as that of log 835107.

In fact, numbers formed of the same digits in the same order differing only in the position of their decimal points, must have their ratios equal to an integral power of 10 and so must have their logarithms differing only by a whole number.

Hence the result.

The two theorems above given show that (i) the characteristic of the logarithm of a number can be found by a simple glance at the number and (ii) that for the mantissa part of the logarithm of a number, we need only take into

account the digits of which the number is formed, without taking any notice of the position of the decimal point in it.

In logarithmic tables, only the mantissæ of the logarithms of numbers are therefore given.

These constitute the special advantages of the common system of logarithms.

78. Examples worked out.

Ex. 1. *Simplify*

$$\log \frac{\sqrt[4]{5}\,.\sqrt[10]{2}}{\sqrt[3]{(18.\sqrt{2})}}, \textit{ and find its value, given}$$

$$\log 2 = \cdot 30103 \textit{ and } \log 3 = \cdot 4771213.$$

$$\text{The given exp.} = \log \frac{5^{\frac{1}{4}}.2^{\frac{1}{10}}}{(18.2^{\frac{1}{2}})^{\frac{1}{3}}}$$

$$= \log \frac{10^{\frac{1}{4}}.2^{\frac{1}{10}}}{2^{\frac{1}{4}}(2.3^2.2^{\frac{1}{2}})^{\frac{1}{3}}} = \log \frac{10^{\frac{1}{4}}.2^{\frac{1}{10}}}{2^{\frac{1}{4}}.2^{\frac{1}{3}}.3^{\frac{2}{3}}.2^{\frac{1}{6}}}$$

$$= \log \frac{10^{\frac{1}{4}}}{2^{\frac{13}{20}}.3^{\frac{2}{3}}} = \log 10^{\frac{1}{4}} - \log\left(2^{\frac{13}{20}} \times 3^{\frac{2}{3}}\right)$$

$$= \tfrac{1}{4} \log 10 - \left(\log 2^{\frac{13}{20}} + \log 3^{\frac{2}{3}}\right)$$

$$= \tfrac{1}{4} \log 10 - \tfrac{13}{20} \log 2 - \tfrac{2}{3} \log 3$$

and its value is

$$\tfrac{1}{4}.1 - \tfrac{13}{20}(\cdot 30103) - \tfrac{2}{3}(\cdot 4771213)$$

$$= \cdot 25 - \cdot 1956695 - \cdot 3180809$$

$$= -1 + \cdot 7362496$$

$$= \bar{1}\cdot 7362496.$$

Note. $\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1 - \log 2$ and hence $\log 5$ is deducible from $\log 2$.

Ex. 2. *Prove that*

$$7 \log \tfrac{10}{9} - 2 \log \tfrac{25}{24} + 3 \log \tfrac{81}{80} = \log 2.$$

The left-hand expression

$$=\log\left(\tfrac{10}{9}\right)^7-\log\left(\tfrac{25}{24}\right)^2+\log\left(\tfrac{81}{80}\right)^3$$

$$=\log\frac{\left(\frac{10}{9}\right)^7\times\left(\frac{81}{80}\right)^3}{\left(\frac{25}{24}\right)^2}$$

$$=\log\left\{\left(\frac{10}{3^2}\right)^7\times\left(\frac{3^4}{10\times2^3}\right)^3\times\left(\frac{3\times2^3\times2^2}{10^2}\right)^2\right\}$$

$$=\log\left(\frac{10^7}{3^{14}}\times\frac{3^{12}}{10^3\times2^9}\times\frac{3^2\times2^{10}}{10^4}\right)$$

$$=\log 2.$$

Alternative method :

Left side

$=7(\log 10-\log 9)-2(\log 25-\log 24)+3(\log 81-\log 80)$

$=7\{\log(5\times2)-\log 3^2\}-2\{\log 5^2-\log(3\times2^3)\}$
$\qquad+3\{\log 3^4-\log(5\times2^4)\}$

$=7\{\log 5+\log 2-2\log 3\}-2\{2\log 5-\log 3-3\log 2\}$
$\qquad+3\{4\log 3-\log 5-4\log 2\}$

$=\log 2.$

Ex. 3. *Find the number of digits in* 4^{15}, *having given log* $2=\cdot30103$.

We have

$$\log 4^{15}=\log 2^{30}=30\log 2$$
$$=30\times\cdot30103=9\cdot0309.$$

Hence, since the characteristic of log 4^{15} is 9, 4^{15} must consist of 10 digits.

Ex. 4. *Find approximately the* 7^{th} *root of* $35\cdot28$, *having given log* $2=\cdot30103$, *log* $3=\cdot4771213$, *log* $7=\cdot8450980$ *and log* $1197\cdot342=3\cdot0782184$.

Let $\quad x=(35\cdot28)^{\frac{1}{7}}=\left(\dfrac{7^2\times3^2\times2^3}{10^2}\right)^{\frac{1}{7}}$

then $\log x=\frac{1}{7}[2\log 7+2\log 3+3\log 2-2\log 10]$

$\qquad=\frac{1}{7}[2\times\cdot8450980+2\times\cdot4771213+3\times\cdot30103-2]$

$\qquad=\cdot0782184$ nearly.

Now, $\log 1197{\cdot}342 = 3{\cdot}0782184$.

$\therefore$ $\log 1{\cdot}197342 = {\cdot}0782184$, having characteristic 0, but mantissa same as that of $\log 1197{\cdot}342$.

Hence, $x = 1{\cdot}197342$ approximately.

Ex. 5. *Obtain an approximate numerical solution of* $2^x.3^{2x} = 100$, *having given* $\log 2 = {\cdot}30103$, $\log 3 = {\cdot}47712$.

We have

$$2^x.3^{2x} = 10^2.$$

$$\therefore \quad \log (2^x.3^{2x}) = \log 10^2,$$

i.e., $x \log 2 + 2x \log 3 = 2 \log 10 = 2$.

$$\therefore \quad x = \frac{2}{\log 2 + 2 \log 3} = \frac{2}{{\cdot}30103 + 2 \times {\cdot}47712}.$$

Note. Equations of this type are called **Exponential equations.**

Examples XIII(a)

[*Use the values :* $\log 2 = {\cdot}30103$, $\log 3 = {\cdot}4771213$, $\log 7 = {\cdot}8450980$ *when required.*]

1. Find the logarithm of (i) 1728 to the base $2\sqrt{3}$, (ii) $\cos^3 a$ to the base $\sec a$.

2. Find $\log_{10} 10000$.

3. Show that $\log_{10} 2$ lies between $\frac{1}{4}$ and $\frac{1}{3}$.

[*C. U. 1926*]

4. Prove that

(i) $\log_a m \times \log_b n = \log_b m \times \log_a n$.

(ii) $\log_2 \log_2 \log_2 16 = 1$.

5. If $\log_e m + \log_e n = \log_e (m+n)$, find m as a simple function of n.

6. Prove that if a series of numbers be in G.P., their logarithms are in A.P.

7. Prove that

$$2 \log a + 2 \log a^2 + 2 \log a^3 + \cdots\cdots + 2 \log a^n$$
$$= n\,(n+1) \log a.$$

8. If x is positive and less than unity, show that

$$\log (1+x) + \log (1+x^2) + \log (1+x^4) + \log (1+x^8) + \cdots \text{ to } \infty$$
$$= -\log (1-x).$$

9. Simplify

(i) $\log_2 \sqrt{6} + \log_2 \sqrt{\tfrac{2}{3}}$.

(ii) $\dfrac{\log \sqrt{27} + \log 8 - \log \sqrt{1000}}{\log 1{\cdot}2}$.

10. Find $\log ({\cdot}0025)^{\frac{1}{3}}$ and $\log (\tfrac{5}{72})^{-\frac{1}{3}}$.

11. Prove that

(i) $\log_a b \times \log_b c \times \log_c a = 1$.

(ii) $\log_a x = \log_b x \times \log_c b \times \log_d c \ldots \times \log_n m \times \log_a n$.

12. Show that

(i) $7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80} = \log 2$.

(ii) $7 \log \frac{15}{16} + 6 \log \frac{8}{3} + 5 \log \frac{2}{5} + \log \frac{32}{25} = \log 3$.

13. Extract the fifth root of 84, having given

$$\log 2425805 = 6{\cdot}3848559.$$

14. Calculate $({\cdot}0020736)^{\frac{1}{7}}$, having given

$$\log 41369 = 4{\cdot}6166750.$$

15. Simplify

(i) $\log \sqrt[7]{\dfrac{8^{\frac{1}{3}} \times 14^{\frac{1}{3}}}{\sqrt{72} \times \sqrt[5]{60}}}$.

(ii) $\sqrt[3]{\dfrac{7{\cdot}2 \times 6{\cdot}3}{62{\cdot}5}}$, having given

$$\log 898665 = 5{\cdot}9535977.$$

16. Find the value of $64\{1-(1\cdot05)^{-20}\}$, having given

$$\log 24121 = 4\cdot382394.$$

17. Find the number of digits in (i) 2^{40}, (ii) 3^{11}, (iii) $(540)^9$.

18. Find the number of zeroes after the decimal point before the first significant digit in the expressions

(i) $(\cdot024)^{15}$. (ii) $\left(\frac{1}{4\cdot05}\right)^8$. (iii) $\left(\cdot0\dot{2}5\dot{9}\right)^{50}$.

19. Solve the equations :—

(i) $3^x = 2$. (ii) $3^{x-4} = 7$.

(iii) $5^{6x} \div 7^{x+2} = 3^{2x-3}$.

(iv) $\left.\begin{array}{l} 2^x = 3^y \\ 2^{y+1} = 3^{x-1} \end{array}\right\}$ (v) $\left.\begin{array}{l} 7^{x+y} \times 3^{2x+y} = 9 \\ 3^{x-y} \div 2^{x-2y} = 3^x \end{array}\right\}$

20. (i) If $\log(x^2y^3) = a$, $\log\left(\frac{x}{y}\right) = b$, find $\log x$ and $\log y$.

(ii) If $a^2 + b^2 = 7ab$, show that

$$\log\{\tfrac{1}{3}(a+b)\} = \tfrac{1}{2}(\log a + \log b).$$

21. If $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$, show that $x^x y^y z^z = 1$.

22. Why is $\log(1+2+3) = \log 1 + \log 2 + \log 3$?

23. If $a, b, c, \ldots\ldots$ be in G.P., show that

$$\log_a x, \log_b x, \log_c x \ldots\ldots\ldots\ldots \text{ are in H.P.}$$

24. If $xy^{l-1} = a$, $xy^{m-1} = b$, $xy^{n-1} = c$, prove that

$$(m-n)\log a + (n-l)\log b + (l-m)\log c = 0.$$

25. If $\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z}$, show that

$$y^z z^y = z^x x^z = x^y y^x.$$

79. Tables of Logarithms and Trigonometrical ratios.

Several mathematical tables correct up to five places of decimals are given at the end of the book. An explanation of the tables is given below.

Table I gives the common logarithm of all numbers from 1 to 10000, *i.e.*, those which consist of 4 digits or less. The tabulated quantities are the mantissæ only, correct to five places, with the decimal point dropped. The characteristic is to be supplied according to the rule given in Art. 77. The main body of the table gives logarithms (mantissa part) of numbers of 3 digits, and the mean difference table at the side supplies the increment in the mantissa due to the fourth digit. This increment is written, in order to save space, giving the significant digits only, which are to be supplied with the necessary number of zeroes to make up 5 places (here the table being a five-figure table). Thus, ·00024 will be written as 24 only in the difference table. As an example, to find log 2·697, we notice from the table that the mantissa for log 269 is ·42975, and along the same row, the difference table gives 115 under the heading 7. This means that for 7 in the fourth place of the number (*i.e.*, for the number 2697) the increment in the mantissa will be ·00115. Hence, log 2697 will have its mantissa ·42975 + ·00115 = ·43090. Again, log 2·697 has the same mantissa but its characteristic is 0. Thus, log 2·697 = 0·43090.

Table II gives ordinary sines and cosines (usually referred to as *natural sines and cosines*) of all angles from 0° to 90° at intervals of $1'$, sines being given from the left side of the top towards the right and downwards, and cosines being given from the right side of the bottom towards the left and upwards. The table is arranged in such a way that the sine of any angle given is the same as the cosine of exactly the complementary angle, and it is on this arrangement that a single table serves as a sine as well as a cosine table. The main portion of the table gives sines or cosines of angles at intervals of $10'$, and the difference

table at the side gives changes in the value of the sine or cosine for changes in minutes in the angles. It should be remembered that as an angle increases from 0° to 90°, its sine increase from 0 to 1 wherease its cosine decreases from 1 to 0. Hence, *the changes given in the difference table are to be added in case of sines and subtracted in case of cosines* for the increased number of minutes in the angles. Moreover, as in Table I, the numbers in the difference table are to be made up to five places of decimals by supplying the requisite number of zeroes before it. For example, using the table, sin 53° 23′ = ·80212 + ·00052 = ·80264 and cos 20° 42′ = ·86892 − ·00029 = ·86863.

Table III similarly gives natural tangents and cotangents of angles from 0° to 90°, obtained at intervals of 1′ with the help of the difference table. The quantities in the difference table, being made up into five figures, are to be *added in case of tangents and subtracted in case of cotangents* for increased number of minutes in the angle.

Table IV gives logarithmic sines and logarithmic cosines of all angles from 0° to 90° at intervals of 1′ (with the aid of the difference table). Logarithmic sine of angle θ, written as **L sin θ** means **10 + log sin θ**, and similarly, logarithmic cosine of θ, written as **L cos θ** means **10 + log cos θ.** In taking logarithms of trigonometrical ratios of angles, it may be noted that sines and cosines of angles are numerically less than unity, and tangents of angles between 0° and 45° as also cotangents of angles between 45° and 90° are less than unity. Hence, logarithms of these quantities are negative. To avoid using negative values in the tables, *logarithms of trigonometrical ratios are always tabulated after adding* 10 *to them.* Thus, the table gives L sin θ and L cos θ (and not log sin θ and log cos θ).

Table V gives logarithmic tangents (*i.e.*, L tan $\theta = 10 +$ log tan θ) and logarithmic cotangents (*i.e.*, L cot $\theta = 10 +$ log cot θ) of all angles from 0° to 90°, obtained at intervals of 1′ with the aid of the difference table.

80. Principle of Proportional Parts.

Suppose we find from table I the logarithms of the two numbers 6257 and 6258, and we want to find the logarithm of 6257·6, or that we find from table III, tan $53^\circ 23'$ and tan $53^\circ 24'$, but we want to find tan $53^\circ 23' 20''$, or similarly, from table IV, we get L cos $37^\circ 42'$ and L cos $37^\circ 43'$ but we want to find L cos $37^\circ 42' 48''$ how are we to proceed?

In order to meet such cases, the 'Principle of Proportional Parts' may be used. The principle may be stated as follows.

If the value of a quantity depending on a variable quantity x be tabulated for different values of x at regular small intervals, then in most cases, for a very small change in x (which is called the argument) the corresponding small change in the tabulated quantity (called the function of the argument) is proportional to the change in x.

We shall assume the truth of this principle, for a strict proof of it, with the proper restriction under which it is true, depends on the use of Calculus. For the tables with which we are concerned, it is true for all practical purposes.

The application of the principle is illustrated in the following examples.

Ex. 1. *Given* $\log 63374 = 4{\cdot}8019111$ *and* $\log 63375 = 4{\cdot}8019180$, *find* $\log 63{\cdot}3743$ *and find the number whose logarithm is* $\bar{2}{\cdot}8019136$.

Here $\quad \log 63375 = 4{\cdot}8019180$

and $\quad \log 63374 = 4{\cdot}8019111$

Hence, for an increase of 1 in the number, the increment in the logarithm is $\cdot0000069$. (This is usually spoken as 'diff. for 1 is 69')

Therefore, by the Principle of Proportional Parts, increase in the logarithm for an increase of $\cdot3$ in the number is

$$\cdot3 \times \cdot0000069 = \cdot00000207$$
$$= \cdot0000021, \text{ up to seven places.}$$

Hence, $\log 63374\cdot3 = 4\cdot8019111 + \cdot0000021$
$$= 4\cdot8019132.$$

$\therefore \quad \log 63\cdot3743 = \bar{1}\cdot8019132.$

Again, $4\cdot8019136$ lies between $4\cdot8019111$ and $4\cdot8019180$, the difference from the former being $\cdot0000025$. Hence, $4\cdot8019136$ is the logarithm of a number lying between 63374 and 63375, say logarithm of $63374 + x$.

Then, diff. for 1 being 69 (*i.e.*, $\cdot0000069$) and diff. for x being 25, (*i.e.*, $\cdot0000025$), by the Principle of Proportional Parts, we have

$$69 : 25 :: 1 : x$$
$$\text{or, } x = \tfrac{25}{69} = \cdot36\cdots\cdots$$

Hence, $\log 63374\cdot36\cdots = 4\cdot8019136$.

The required number whose logarithm is $\bar{2}\cdot8019136$, having the same mantissa, must be formed of the same digits arranged in the same order, and its characteristic being -2, the number must be $\cdot0633743 6\cdots$

Ex. 2. (i) *Given* $L \sin 37° \, 43' \, 50'' = 9\cdot7867152$
$L \sin 37° \, 44' = 9\cdot7867424$,
find $L \sin 37° \, 43' \, 56''$.

(ii) *Given* $L \tan 79° \, 51' \, 40'' = 10\cdot7475657$
$L \tan 79° \, 51' \, 50'' = 10\cdot7476872$,
find the angle whose $L \tan$ *is* $10\cdot7476532$.

[*C. U. 1921*]

In (i) diff. (in the value of L sin) for $10''$ (diff. in angle)

$= 272$ (*i.e.*, $\cdot0000272$)

hence, diff. for $6'' = \frac{6}{10} \times 272 = 163\cdot2$ *i.e.*, $\cdot00001632$

and so $L \sin 37° \, 43' \, 56'' = 9\cdot7867152 + \cdot0000163$

$= 9\cdot7867315.$

In (ii) the angle whose L tan is $10\cdot7476532$ evidently lies between $79° \, 51' \, 40''$ and $79° \, 51' \, 50''$.

Let the angle be $79° \, 51' \, 40'' + x''$.

Now, diff. (in the value of L tan) for $10''$ (diff. in angle)

$= 1215$ (*i.e.*, $\cdot0001215$)

and diff. for $x'' = 875$

(*i.e.*, $\cdot0000875$, being $10\cdot7476532 - 10\cdot7475657$)

$$\therefore \quad \frac{x}{10} = \frac{875}{1215} \text{ or } x = 7\cdot2 \text{ nearly.}$$

Thus, the required angle is $79° \, 51' \, 47''\cdot2$.

Ex. 3. *Given* $\cos 58° \, 17' = \cdot5257191$ *and diff. for* $1' = 2474$, *find* $\cos 58° \, 17' \, 20''$.

Here, diff. for $1'$ *i.e.*, $60'' = 2474$,

$\therefore$ diff. for $20'' = \frac{20}{60} \times 2474 = 825$ (nearly).

As for increasing angle, cosine diminishes,

$\therefore \cos 58° \, 17' \, 20'' = \cdot5257191 - \cdot0000825$

$= \cdot5256366.$

Examples XIII(b)

1. Given $\log 18\cdot906 = 1\cdot2765997$

and $\log 18\cdot907 = 1\cdot2766226$,

find $\log 1890\cdot635$.

2. Given $\log 69714 = 4\cdot8433200$

$\log 69715 = 4\cdot8433262$,

find $\log (\cdot000697145)^{\frac{1}{3}}$.

3. Given $\log 37602 = 4{\cdot}5752109$
$\log 37601 = 4{\cdot}5751994$,
find the number whose logarithm is $1{\cdot}5752086$.

4. Given $\log 3 = {\cdot}4771213$
$\log 74008 = 4{\cdot}8692787$
diff. for $1' = 59$,
find $({\cdot}09)^{\frac{1}{8}}$.

5. Given $\cos 32^\circ\, 16' = {\cdot}8455726$
and $\cos 32^\circ\, 17' = {\cdot}8454172$,
find the value of $\cos 32^\circ\, 16'\, 21''$
and find the angle whose cosine is ${\cdot}8455176$.

6. Find $\tan 38^\circ\, 24'\, 37{\cdot}5''$, having given
$\tan 38^\circ\, 24' = {\cdot}7925902$ and $\tan 38^\circ\, 25' = {\cdot}7930640$.

7. Given $L \sin 44^\circ\, 17' = 9{\cdot}8439842$
and $L \sin 44^\circ\, 18' = 9{\cdot}8441137$,
find $L \sin 44^\circ\, 17'\, 33''$. Deduce the value of
$L \operatorname{cosec} 44^\circ\, 17'\, 33''$.

8. Given $L \sin 36^\circ\, 24' = 9{\cdot}7733614$
$L \sin 36^\circ\, 25' = 9{\cdot}7735327$,
find the angle whose $L \sin$ is $9{\cdot}7734642$.

9. If $L \cot 53^\circ\, 13' = 9{\cdot}8736937$
$L \cot 53^\circ\, 14' = 9{\cdot}8734302$,
find θ where $L \cot \theta = 9{\cdot}8734523$.

10. Given $L \tan 22^\circ\, 37' = 9{\cdot}6197205$
diff. for $1' = 3557$,
find the value of

$$L \tan 22^\circ\, 37'\, 22''$$

and the angle whose $L \tan$ is $9{\cdot}6195283$.

11. Prove that, θ being any acute angle,

$$L \sin\theta + L \operatorname{cosec}\theta = L \cos\theta + L \sec\theta$$
$$= L \tan\theta + L \cot\theta = 20.$$

12. Given $L \cos 36^\circ\ 40' = 9{\cdot}9042111$, find $L \sec 36^\circ\ 40'$.

13. Given $L \cos 34^\circ\ 44' = 9{\cdot}9147729$

$L \cos 34^\circ\ 45' = 9{\cdot}9146852$,

find the value of $L \cos 34^\circ\ 44'\ 27''$.

14. Given $L \sin 36^\circ\ 40' = 9{\cdot}7760897$

$L \cos 36^\circ\ 40' = 9{\cdot}9042111$,

find $L \tan 36^\circ\ 40'$.

15. Prove that the difference of tabular logarithms of any two ratios is equal to the difference of the logarithms of those two ratios.

16. If $\sin\theta = {\cdot}8$, find θ

given $\log 2 = {\cdot}3010300$

$L \sin 53^\circ\ 7' = 9{\cdot}9030136$

$L \sec 36^\circ\ 52' = 10{\cdot}0968916$.

17. Find the value of

$$\frac{\sin 34^\circ\ 17' \times \cos 77^\circ\ 23'}{\tan 27^\circ\ 12'}$$

given $L \sin 12^\circ\ 37' = 9{\cdot}3393$

$L \cos 55^\circ\ 43' = 9{\cdot}7507$

$L \tan 62^\circ\ 48' = 10{\cdot}2891$

and $\log 23{\cdot}94 = 1{\cdot}3791$.

CHAPTER XIV

PROPERTIES OF TRIANGLES

81. In a triangle ABC, there are six parts, the three sides and the three angles. It is usual to denote the angles of the triangle by A, B, C and the corresponding opposite sides by a, b, c. The six parts are not independent of one another. The various relations existing among them are deduced in the following articles.

82. *In any triangle, prove that*

$$\frac{\mathbf{a}}{\sin \mathbf{A}} = \frac{\mathbf{b}}{\sin \mathbf{B}} = \frac{\mathbf{c}}{\sin \mathbf{C}}.$$

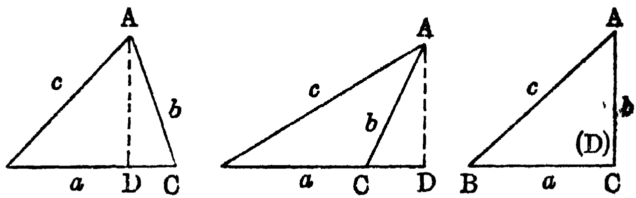


Fig. (i) Fig. (ii) Fig. (iii)

Let ABC be any triangle. From A draw AD perpendicular to BC or BC produced if necessary [*Fig. (ii)*]

[In Fig. (i), C is an *acute* angle, in Fig. (ii), C is an *obtuse* angle, in Fig. (iii), C is a *right angle*.]

From $\triangle ABD$, $AD = AB \sin ABD = c \sin B$.

From $\triangle ACD$, $AD = AC \sin ACD = b \sin C$ [*Fig. (i)*]

or, $= b \sin (\pi - C)$ [*Fig. (ii)*]

i.e., $= b \sin C$.

$$\therefore \quad b \sin C = c \sin B, \qquad i.e., \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Similarly, by drawing a perpendicular from B upon CA, we have $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

In Fig. (iii), C is a *right* angle :

$$\therefore \quad \sin A = \frac{a}{c}; \ \sin B = \frac{b}{c}; \ \sin C = 1.$$

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = c = \frac{c}{\sin C}$$

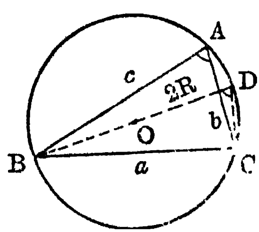
Hence, in all cases,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \quad \cdots \quad (1)$$

Thus, in any triangle,

the sides are proportional to the sines of the opposite angles.

An alternative method of Proof :



Let O be the centre and R be the radius of the circle circumscribing the triangle ABC.

Join BO and produce it to meet the circumference in D. Join CD. The $\angle BCD$ is then a right angle.

From Δ BCD, $\sin BDC = \dfrac{BC}{BD} = \dfrac{a}{2R}$.

But $\angle BDC = \angle A$, being in the same segment.

$$\therefore \quad \frac{a}{2R} = \sin A, \quad \text{or,} \quad \frac{a}{\sin A} = 2R.$$

Similarly, by joining AO and producing it to meet the circumference in E, and joining CE, BE, it can be shown that

$$\frac{b}{\sin B} = 2R \quad \text{and} \quad \frac{c}{\sin C} = 2R.$$

$$\therefore \quad \mathbf{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.} \qquad \cdots \quad (2)$$

Note 1. If angle A be obtuse, A and D fall on opposite sides of BC and $ABCD$ being cyclic, $\sin BDC = \sin (180° - A) = \sin A$, and the same result follows. In case A is a right angle, evidently $2R = a = a/\sin A$, and we get the same result.

Note 2. It follows from the relation (2) that

$$a = 2R \sin A,\ b = 2R \sin B,\ c = 2R \sin C;$$

$$\sin A = \frac{a}{2R},\ \sin B = \frac{b}{2R},\ \sin C = \frac{c}{2R}.$$

83. *In any triangle, to prove that*

$$\mathbf{a^2 = b^2 + c^2 - 2bc \cos A}, \text{ or, } \mathbf{\cos A = \frac{b^2 + c^2 - a^2}{2bc}}.$$

$$\mathbf{b^2 = c^2 + a^2 - 2ca \cos B}, \text{ or, } \mathbf{\cos B = \frac{c^2 + a^2 - b^2}{2ca}}.$$

$$\mathbf{c^2 = a^2 + b^2 - 2ab \cos C}, \text{ or, } \mathbf{\cos C = \frac{a^2 + b^2 - c^2}{2ab}}.$$

Take the figures of Art. 82.

First, let C be an *acute* angle [*Fig.* (*i*)]; then from Geometry,

$$AB^2 = BC^2 + CA^2 - 2BC \,.\, CD.$$

Now, from $\triangle ACD$, $CD = AC \cos C = b \cos C$.

$$\therefore \quad c^2 = a^2 + b^2 - 2ab \cos C.$$

Next, let the angle C be an *obtuse* angle [*Fig.* (*ii*)]; then from geometry,

$$AB^2 = BC^2 + CA^2 + 2BC \,.\, CD.$$

Now, from $\triangle ACD$, $CD = AC \cos ACD$

$$= b \cos(\pi - C) = -b \cos C.$$

$\therefore \quad c^2 = a^2 + b^2 - 2ab \cos C.$

Lastly, let C be a *right* angle [*Fig.* (*iii*)] ; then from Geometry,

$$AB^2 = BC^2 + CA^2,$$

i.e., $\quad c^2 = a^2 + b^2 = a^2 + b^2 - 2ab \cos C.$

[$\because \cos C = \cos 90° = 0.$]

Hence, for all values of C, we have

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

$$\therefore \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Similarly, the other two relations can be established.

Obs. This theorem expresses the cosines of the angles of a triangle in terms of the sides.

84. *In any triangle, to prove that*

$$\mathbf{a = b \cos C + c \cos B.}$$

$$\mathbf{b = c \cos A + a \cos C.}$$

$$\mathbf{c = a \cos B + b \cos A.}$$

Take the figures of Art. 82.

In Fig. (i), where C is an *acute* angle,

$$BC = BD + CD$$

$$= AB \cos ABD + AC \cos ACD.$$

$\therefore \quad a = c \cos B + b \cos C.$

In Fig. (ii), where C is an *obtuse* angle,

$$BC = BD - CD$$

$$= AB \cos ABD - AC \cos ACD$$

$$= c \cos B - b \cos(180° - C)$$

$$= c \cos B + b \cos C.$$

In Fig. (iii), where C is a *right* angle,

$$BC = AB \cos B.$$

$\therefore \quad a = c \cos B = c \cos B + b \cos C.$

$[\because \cos C = \cos 90° = 0.]$

Thus, in all cases,

$$a = b \cos C + c \cos B.$$

Similarly, the other two relations can be established.

85. From Art. 83 and note of Art. 82, it follows that

$$\tan A = \frac{\sin A}{\cos A} = \frac{\dfrac{a}{2R}}{\dfrac{b^2 + c^2 - a^2}{2bc}} = \frac{abc}{R} \cdot \frac{1}{b^2 + c^2 - a^2}.$$

Similarly, $\tan B = \dfrac{abc}{R} \cdot \dfrac{1}{c^2 + a^2 - b^2}$;

$$\tan C = \frac{abc}{R} \cdot \frac{1}{a^2 + b^2 - c^2}.$$

86. Trigonometrical ratios of half angles of a triangle in terms of the sides.

We have, $2 \sin^2 \dfrac{A}{2} = 1 - \cos A = 1 - \dfrac{b^2 + c^2 - a^2}{2bc}$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (b^2 - 2bc + c^2)}{2bc}$$

$$= \frac{a^2 - (b - c)^2}{2bc} = \frac{(a - b + c)(a + b - c)}{2bc}.$$

Let s denote the semi-perimeter of the triangle :

then $\quad 2s = a + b + c.$

Now, $\quad a - b + c = a + b + c - 2b = 2s - 2b = 2(s - b),$

$\quad a + b - c = a + b + c - 2c = 2s - 2c = 2(s - c).$

Hence, $2 \sin^2 \dfrac{A}{2} = \dfrac{2(s - b) \cdot 2(s - c)}{2bc}$

i.e., $\quad \sin^2 \dfrac{A}{2} = \dfrac{(s - b)(s - c)}{bc},$

$\therefore \quad \sin \dfrac{A}{2} = \sqrt{\dfrac{(s - b)(s - c)}{bc}}.$

The positive value of the square root must be taken; for A, being an angle of a triangle, is less than 180°; and hence, $\frac{1}{2}A < 90^\circ$ and consequently, $\sin \frac{1}{2}A$ must always be positive.

Again, $2\cos^2 \dfrac{A}{2} = 1 + \cos A$

$$= 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b+c)^2 - a^2}{2bc} = \frac{(b+c+a)(b+c-a)}{2bc}.$$

Now, $\quad b + c - a = a + b + c - 2a = 2s - 2a = 2(s - a)$.

$$\therefore \quad 2\cos^2 \frac{A}{2} = \frac{2s.2(s-a)}{2bc}, \ i.e., \cos^2 \frac{A}{2} = \frac{s(s-a)}{bc}.$$

$$\therefore \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

Here also the positive value of the square root must be taken; for $\frac{1}{2}A$ being less than 90°, $\cos \frac{1}{2}A$ is always positive.

Again, $\tan \dfrac{A}{2} = \sin \dfrac{A}{2} \div \cos \dfrac{A}{2}$

$$= \sqrt{\frac{(s-b)(s-c)}{bc}} \div \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Similarly, the trigonometrical ratios of $\dfrac{B}{2}, \dfrac{C}{2}$ can be obtained in terms of the sides.

Note. *Without assuming the values of $\sin \frac{1}{2}A$, $\cos \frac{1}{2}A$*, the value of $\tan \frac{1}{2}A$ can be obtained by substituting the value of $\cos A$ in terms of the sides from Art. 83 in the relation $\tan^2 \frac{1}{2}A = \dfrac{1-\cos A}{1+\cos A}$ and then extracting the square root after simplification.

Thus, we have

$$\left.\begin{aligned} \sin\frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}} \\ \sin\frac{B}{2} &= \sqrt{\frac{(s-c)(s-a)}{ca}} \\ \sin\frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}} \end{aligned}\right\} \quad \cdots \quad \cdots (1)$$

$$\left.\begin{aligned} \cos\frac{A}{2} &= \sqrt{\frac{s(s-a)}{bc}} \\ \cos\frac{B}{2} &= \sqrt{\frac{s(s-b)}{ca}} \\ \cos\frac{C}{2} &= \sqrt{\frac{s(s-c)}{ab}} \end{aligned}\right\} \quad \cdots \quad \cdots (2)$$

$$\left.\begin{aligned} \tan\frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \\ \tan\frac{B}{2} &= \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \\ \tan\frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \end{aligned}\right\} \quad \cdots \quad \cdots (3)$$

87. Sine of an angle of a triangle in terms of the sides.

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$= 2\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{s(s-a)}{bc}}. \quad [\textit{Art. 86}]$$

$$\therefore \quad \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

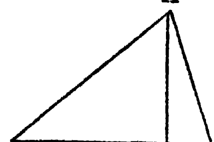
Similarly, $\sin B = \dfrac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)}.$

$$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}.$$

$\sqrt{s(s-a)(s-b)(s-c)}$, being the expression for the area of the triangle [See *Art. 88*], is usually denoted by the Greek letter Δ. Hence, the above formulæ may be written as

$$\sin \mathbf{A} = \frac{2\Delta}{\mathbf{bc}}, \ \sin \mathbf{B} = \frac{2\Delta}{\mathbf{ca}}, \ \sin \mathbf{C} = \frac{2\Delta}{\mathbf{ab}}.$$

88. Area of a triangle.



Let ABC be a triangle and let Δ denote its area. Draw AD perpendicular to BC, then from ΔACD,

$$AD = AC \sin C = b \sin C.$$

Now, $\quad \Delta = \frac{1}{2} BC.AD = \frac{1}{2} ab \sin C.$

Similarly by drawing perpendicular from B and C to the opposite sides, it can be shown that

$$\Delta = \tfrac{1}{2} bc \sin A = \tfrac{1}{2} ca \sin B.$$

Otherwise, $\quad \Delta = \frac{1}{2} ab \sin C$

$$= \tfrac{1}{2} ca \sin B \ [\because b \sin C = c \sin B]$$
$$= \tfrac{1}{2} bc \sin A \ [\because a \sin B = b \sin A]$$

Thus, $\quad \boldsymbol{\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \frac{1}{2} ab \sin C} \qquad \cdots \quad$ (i)

$= \frac{1}{2}$(*product of two sides*) $\times$ (*sine of included angle*).

Again, $\quad \Delta = \frac{1}{2} bc \sin A = bc \sin \frac{A}{2} \cos \frac{A}{2}$

$$= bc \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}}$$

$$= \boldsymbol{\sqrt{s(s-a)(s-b)(s-c)}}. \qquad \cdots \qquad \cdots \quad \text{(ii)}$$

Substituting in the expression $s=\frac{1}{2}(a+b+c)$, we get

$$\Delta=\frac{1}{4}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}$$

$$=\frac{1}{4}\{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4\}^{\frac{1}{2}} \quad \cdots \quad \text{(iii)}$$

Again,

$$\Delta=\frac{1}{2}bc\sin A=\frac{1}{2}bc\cdot\frac{a}{2R}\,[\,Art.\ 82\,]=\frac{abc}{4R} \quad \cdots \quad \text{(iv)}$$

Note. In some text books, S is used to denote the area of a triangle, but to avoid confusion between S and s in writing, the symbol Δ is preferable.

89. *In any triangle, to prove that*

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}.$$

We have, in any triangle,

$$\frac{b}{c}=\frac{\sin B}{\sin C}.$$

$$\therefore \quad \frac{b-c}{b+c}=\frac{\sin B-\sin C}{\sin B+\sin C}=\frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$=\cot\frac{B+C}{2}\tan\frac{B-C}{2}$$

$$=\tan\frac{A}{2}\tan\frac{B-C}{2}\left[\because \frac{A}{2}+\frac{B+C}{2}=90^\circ\right]$$

$$\therefore \quad \tan\frac{B-C}{2}=\frac{b-c}{b+c}\cdot\frac{1}{\tan\frac{A}{2}}=\frac{b-c}{b+c}\cot\frac{A}{2}.$$

Similarly,

$$\tan\frac{C-A}{2}=\frac{c-a}{c+a}\cot\frac{B}{2};\ \tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

90. The three sets of formulæ in Arts. 82, 83, 84 have been established directly from the figures. These three sets

however, are not independent, for, from any one set, the other two sets can be deduced.

For example, let us deduce the formulæ of Art. 83 from those of Art. 84.

By Art. 84,
$$a=b\cos C+c\cos B$$
$$b=c\cos A+a\cos C$$
$$c=a\cos B+b\cos A.$$

Multiplying these in succession by a, b and c, and subtracting the first result from the sum of the other two, we have,

$$b^2+c^2-a^2=b(c\cos A+a\cos C)+c(a\cos B+b\cos A)$$
$$-a(b\cos C+c\cos B)=2bc\cos A.$$

$\therefore\quad \cos A=\dfrac{b^2+c^2-a^2}{2bc}$; similarly, for $\cos B$, $\cos C$.

Note. For other cases, see *Appendix*.

91. In working out identities which involve both the sides and angles of a triangle, it is sometimes convenient to express the sides in terms of the angles, or the angles in terms of the sides.

Also, it is sometimes found convenient to express the values of $\tan\dfrac{A}{2}$, $\tan\dfrac{B}{2}$, $\tan\dfrac{C}{2}$ in a form in which the denominator is constant and numerator is free from radical. Thus, multiplying the numerator and the denominator of the value of $\tan\dfrac{A}{2}$ by $\sqrt{(s-b)(s-c)}$ and noting that

$\sqrt{s(s-a)(s-b)(s-c)}=\Delta$, we have

$\tan\dfrac{A}{2}=\dfrac{(s-b)(s-c)}{\Delta}$; similarly, $\tan\dfrac{B}{2}=\dfrac{(s-c)(s-a)}{\Delta}$;

$\tan\dfrac{C}{2}=\dfrac{(s-a)(s-b)}{\Delta}$.

Again, multiplying the numerator and the denominator of the value of $\cot \frac{A}{2}$ by $\sqrt{s(s-a)}$, we have

$$\cot \frac{A}{2} = \frac{s(s-a)}{\Delta}.$$

Similarly, $\cot \frac{B}{2} = \frac{s(s-b)}{\Delta}$; $\cot \frac{C}{2} = \frac{s(s-c)}{\Delta}$.

Ex. 1. *Show that in any triangle,*

$$a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0.$$

Left side $= (a \sin B - b \sin A) + (b \sin C - c \sin B)$
$\qquad + (c \sin A - a \sin C)$

$$= 0 + 0 + 0 \left[\because \text{ by Art. 82, } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \right]$$

$= 0.$

Ex. 2. *Show that in any triangle,*

$$a \sin(B - C) + b \sin(C - A) + c \sin(A - B) = 0.$$

[*H. S. 1961*]

$a = 2R \sin A$ [*by Art. 82*] $= 2R \sin(B + C)$, $[\because A + B + C = \pi]$

$\therefore$ 1st term of the left side $= 2R \sin(B + C) \sin(B - C)$
$\qquad = 2R(\sin^2 B - \sin^2 C)$.

[*by Ex. 2, Art. 85*]

Similarly, 2nd term $= 2R(\sin^2 C - \sin^2 A)$

3rd term $= 2R(\sin^2 A - \sin^2 B)$.

Now adding together the three terms, the required result follows.

Ex. 3. *In any triangle, prove that*

$$(b - c) \cot \frac{A}{2} + (c - a) \cot \frac{B}{2} + (a - b) \cot \frac{C}{2} = 0.$$

Substituting the values of $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$, as given in Art. 91, we have, the left side

$$=(b-c)\cdot\frac{s(s-a)}{\Delta}+(c-a)\cdot\frac{s(s-b)}{\Delta}+(a-b)\cdot\frac{s(s-c)}{\Delta}$$

$$=\frac{s}{\Delta}\left[(b-c)(s-a)+(c-a)(s-b)+(a-b)(s-c)\right]$$

$$=\frac{s}{\Delta}\cdot 0=0.$$

Ex. 4. *If the cosines of two of the angles of a triangle are inversely proportional to the opposite sides, show that the triangle is either isosceles or right-angled.*

We have, by the question,

$$\frac{\cos A}{\cos B}=\frac{b}{a}=\frac{\sin B}{\sin A}. \qquad [\textit{by Art. 82}]$$

$\therefore$ $\sin A\cos A=\sin B\cos B$, or, $\sin 2A=\sin 2B$,

or, $\sin 2A-\sin 2B=0$,

or, $2\cos(A+B)\sin(A-B)=0$.

$\therefore$ either $\cos(A+B)=0$, *i.e.*, $(A+B)=90°$,

i.e., the triangle is right-angled ;

or, $\sin(A-B)=0$, *i.e.*, $A-B=0$, *i.e.*, $A=B$,

i.e., the triangle is isosceles.

Ex. 5. *If the sides of a triangle are in A. P., show that* $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ *are also in A. P.*

$\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ are in A. P.,

if $\cot\frac{B}{2}-\cot\frac{A}{2}=\cot\frac{C}{2}-\cot\frac{B}{2}$,

i.e., if $\dfrac{s(s-b)}{\Delta} - \dfrac{s(s-a)}{\Delta} = \dfrac{s(s-c)}{\Delta} - \dfrac{s(s-b)}{\Delta}$,

i.e., if $(s-b)-(s-a)=(s-c)-(s-b)$,

i.e., if $a-b=b-c$,

i.e., if a, b, c are in A.P.

Ex. 6. *Show that*

$$b^2 \sin 2C + c^2 \sin 2B = 4\Delta.$$

Left side $= b^2.2 \sin C \cos C + c^2.2 \sin B \cos B$

$= 2b \sin C.b \cos C + 2c \sin B.c \cos B$

$= 2b \sin C\,(b \cos C + c \cos B)$

[$\because$ $c \sin B = b \sin C$]

$= 2ab \sin C$ [*by Art. 84*]

$= 4.\frac{1}{2}ab \sin C = 4\Delta.$ [*by Art. 68*]

Examples XIV(a)

In any triangle, prove that (*Ex. 1 to 21*) :—

1. $\sin \dfrac{B-C}{2} = \dfrac{b-c}{a} \cos \dfrac{A}{2}$. [*H. S. 1961 Comp.*]

2. $\cos \dfrac{B-C}{2} = \dfrac{b+c}{a} \sin \dfrac{A}{2}$. [*H. S. 1961 Comp.*]

3. $(b+c) \cos A + (c+a) \cos B + (a+b) \cos C = a+b+c.$

4. $\dfrac{a+b}{a-b} = \tan \dfrac{A+B}{2} \cot \dfrac{A-B}{2}.$

5. $a^2+b^2+c^2 = 2(bc \cos A + ca \cos B + ab \cos C).$

6. $(b+c-a) \tan \dfrac{A}{2} = (c+a-b) \tan \dfrac{B}{2} = (a+b-c) \tan \dfrac{C}{2}.$

7. $\frac{a\sin(B-C)}{b^2-c^2}=\frac{b\sin(C-A)}{c^2-a^2}=\frac{c\sin(A-B)}{a^2-b^2}$.

8. $a^2(\sin^2 B-\sin^2 C)+b^2(\sin^2 C-\sin^2 A)$
$+c^2(\sin^2 A-\sin^2 B)=0$.

9. $a^2(\cos^2 B-\cos^2 C)+b^2(\cos^2 C-\cos^2 A)$
$+c^2(\cos^2 A-\cos^2 B)=0$.

10. $\frac{a^2\sin(B-C)}{\sin B+\sin C}+\frac{b^2\sin(C-A)}{\sin C+\sin A}+\frac{c^2\sin(A-B)}{\sin A+\sin B}=0$.

11. $a\sin\frac{A}{2}\sin\frac{B-C}{2}+b\sin\frac{B}{2}\sin\frac{C-A}{2}$
$+c\sin\frac{C}{2}\sin\frac{A-B}{2}=0$.

12. $\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0$.

13. $a^3\sin(B-C)+b^3\sin(C-A)+c^3\sin(A-B)=0$.

14. $a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)=3abc$.

15. $\frac{a^2\sin(B-C)}{\sin A}+\frac{b^2\sin(C-A)}{\sin B}+\frac{c^2\sin(A-B)}{\sin C}=0$.

16. $(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0$.

17. $\frac{b^2-c^2}{\cos B+\cos C}+\frac{c^2-a^2}{\cos C+\cos A}+\frac{a^2-b^2}{\cos A+\cos B}=0$.

18. $(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}$.

19. $\frac{b-c}{a}\cos^2\frac{A}{2}+\frac{c-a}{b}\cos^2\frac{B}{2}+\frac{a-b}{c}\cos^2\frac{C}{2}=0$.

20. $bc\cos^2\frac{A}{2}+ca\cos^2\frac{B}{2}+ab\cos^2\frac{C}{2}=s^2$.

21. $\frac{1}{a}\cos^2\frac{A}{2}+\frac{1}{b}\cos^2\frac{B}{2}+\frac{1}{c}\cos^2\frac{C}{2}=\frac{s^2}{abc}$.

22. If A be 60°, show that $b+c=2a\cos\frac{B-C}{2}$.

23. Show that a triangle having its sides equal to 3, 5, 7 is an obtuse-angled triangle and determine the obtuse angle.

24. Given $(a+b+c)(b+c-a)=3bc$, find A.

25. If $c^4-2(a^2+b^2)c^2+a^4+a^2b^2+b^4=0$, prove that

$$C=60°, \text{ or, } 120°.$$

26. If $a^4+b^4+c^4=2c^2(a^2+b^2)$, prove that

$$C=45°, \text{ or, } 135°.$$

27. The sides of a triangle are $2x+3$, x^2+3x+3, x^2+2x; show that the greatest angle is 120°.

28. If $\dfrac{1}{a+c}+\dfrac{1}{b+c}=\dfrac{3}{a+b+c}$, show that $C=60°$.

29. If $a=2b$ and $A=3B$, find the angles of the triangle.

30. If the cosines of two of the angles of a triangle are proportional to the opposite sides, show that the triangle is isosceles.

31. If $\cos A=\dfrac{\sin B}{2\sin C}$, show that the triangle is isosceles.

32. If $(a^2+b^2)\sin(A-B)=(a^2-b^2)\sin(A+B)$, prove that the triangle is either isosceles or right-angled.

33. If $(\cos A+2\cos C):(\cos A+2\cos B)=\sin B:\sin C$, prove that the triangle is either isosceles or right-angled.

34. If a^2, b^2, c^2 be in A.P., prove that $\cot A$, $\cot B$, $\cot C$ are also in A P.

35. If $a\cos^2\dfrac{C}{2}+c\cos^2\dfrac{A}{2}=\dfrac{3b}{2}$, show that the sides of the triangle are in A P.

36. If $\sin A:\sin C=\sin(A-B):\sin(B-C)$, show that a^2, b^2, c^2 are in A.P.

37. If a, b, c are in A.P., show that
$\cos A \cot \frac{1}{2}A, \cos B \cot \frac{1}{2}B, \cos C \cot \frac{1}{2}C$ are in A.P.
[$\cos A \cot \frac{1}{2}A = (1-2\sin^2 \frac{1}{2}A)\cot \frac{1}{2}A = \cot \frac{1}{2}A - \sin A$.]

38. Assuming $\Delta = \frac{1}{2}bc \sin A$ and using the value of $\cos A$ in terms of sides, show that

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

39. Find the area of the triangle whose sides are

$$\frac{y}{z}+\frac{z}{x},\ \frac{z}{x}+\frac{x}{y},\ \frac{x}{y}+\frac{y}{z}.$$

40. In a triangle, if $a=13$, $b=14$, $c=15$, find its area.

Prove that in any triangle :

41. $\frac{a^2-b^2}{2} \cdot \frac{\sin A \sin B}{\sin (A-B)} = \Delta.$

42. $4\Delta(\cot A + \cot B + \cot C) = a^2+b^2+c^2.$

43. $a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C.$

44. $a \sin B \sin C + b \sin C \sin A + c \sin A \sin B = \frac{3\Delta}{R}.$

45. $(a \sin A + b \sin B + c \sin C)^2$
$$= (a^2+b^2+c^2)(\sin^2 A + \sin^2 B + \sin^2 C).$$

46. $\frac{\cos B \cos C}{bc} + \frac{\cos C \cos A}{ca} + \frac{\cos A \cos B}{ab} = \frac{1}{4R^2}.$

[*Use* $\Sigma \cot B \cot C = 1$; *ex.* 2, *Ex.* X.]

47. $\frac{b^2-c^2}{a} \cos A + \frac{c^2-a^2}{b} \cos B + \frac{a^2-b^2}{c} \cos C = 0.$

48. $\frac{\cos A}{a} + \frac{a}{bc} = \frac{\cos B}{b} + \frac{b}{ca} = \frac{\cos C}{c} + \frac{c}{ab}.$

49. $4\Delta = a^2 \cot A + b^2 \cot B + c^2 \cot C.$

50. $\left(\frac{a^2}{\sin A} + \frac{b^2}{\sin B} + \frac{c^2}{\sin C}\right) \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \Delta.$

92. Circum-radius of a triangle.

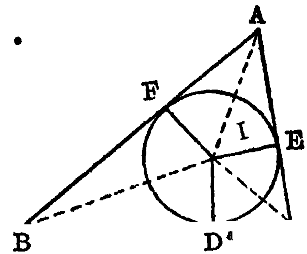
From Art. 82, we have

$$\frac{\mathbf{a}}{\sin \mathbf{A}} = \frac{\mathbf{b}}{\sin \mathbf{B}} = \frac{\mathbf{c}}{\sin \mathbf{C}} = \mathbf{2R}. \qquad \cdots \quad \text{(i)}$$

Hence, $\mathbf{R} = \dfrac{a}{2 \sin A} = \dfrac{abc}{2bc \sin A} = \dfrac{\mathbf{abc}}{\mathbf{4\Delta}}. \qquad \cdots \quad \text{(ii)}$

93. In-radius of a triangle.

Let I be the centre and r the radius of the circle inscribed in the triangle ABC; let D, E, F be the points of contact of the in-circle with the sides BC, CA, AB respectively.



Then, $ID = IE = IF = r$.

Join IA, IB, IC.

$$\begin{aligned}\triangle ABC &= \triangle IBC + \triangle ICA + \triangle IAB \\ &= \tfrac{1}{2}BC.ID + \tfrac{1}{2}CA.IE + \tfrac{1}{2}AB.IF \\ &= \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr \\ &= \tfrac{1}{2}r(a+b+c) = rs.\end{aligned}$$

Thus, $\quad \Delta = rs$.

$$\therefore \quad \mathbf{r} = \frac{\Delta}{\mathbf{s}}. \qquad \cdots \quad \text{(i)}$$

Again, $\quad a = BC = BD + DC$

$= r \cot \tfrac{1}{2}B + r \cot \tfrac{1}{2}C, \quad$ from Δ^s IBD, ICD,

$$= r\left[\frac{\cos \tfrac{1}{2}B}{\sin \tfrac{1}{2}B} + \frac{\cos \tfrac{1}{2}C}{\sin \tfrac{1}{2}C}\right]$$

$$= r\left[\frac{\cos \tfrac{1}{2}B \sin \tfrac{1}{2}C + \sin \tfrac{1}{2}B \cos \tfrac{1}{2}C}{\sin \tfrac{1}{2}B \sin \tfrac{1}{2}C}\right]$$

$$= r\frac{\sin(\tfrac{1}{2}B + \tfrac{1}{2}C)}{\sin \tfrac{1}{2}B \sin \tfrac{1}{2}C} = r\frac{\cos \tfrac{1}{2}A}{\sin \tfrac{1}{2}B \sin \tfrac{1}{2}C}.$$

[$\because \tfrac{1}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C = 90^\circ$, $\sin(\tfrac{1}{2}B + \tfrac{1}{2}C) = \sin(90^\circ - \tfrac{1}{2}A) = \cos \tfrac{1}{2}A$.]

$$\therefore \quad r = a \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \sec \tfrac{1}{2}A = a \frac{\sin \frac{1}{2}B \sin \frac{1}{2}C}{\cos \frac{1}{2}A}.$$

Since, by Art. 92(i), $a = 2R \sin A = 4R \sin \frac{1}{2}A \cos \frac{1}{2}A$,

$$\therefore \quad \mathbf{r = 4R \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C.} \quad \cdots \quad \text{(ii)}$$

Since, from the figure, $AF = AE$, $BD = BF$, $CD = CE$ and since, the sum of these six quantities equal to the perimeter,

$$\therefore \quad AF + BD + CD = \text{semi-perimeter} = s,$$

$$i.e., \quad AF + BC, \quad \text{or}, \quad AF + a = s,$$

$$\therefore \quad AF = s - a = AE.$$

Similarly, $BF = s - b = BD$; $CE = s - c = CD$.

From $\triangle AIF$, $IF = AF \tan IAF$.

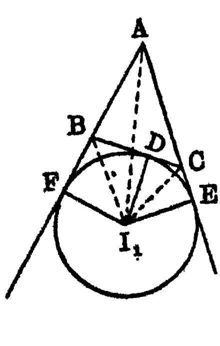
$$\left.\begin{aligned} \therefore \quad & \mathbf{r = (s-a) \tan \tfrac{1}{2}A.} \\ \text{Similarly,} \quad & \mathbf{r = (s-b) \tan \tfrac{1}{2}B,} \\ \text{and} \quad & \mathbf{r = (s-c) \tan \tfrac{1}{2}C.} \end{aligned}\right\} \quad \cdots \quad \text{(iii)}$$

Note. *Distances of the in-centre from the vertices.*

From $\triangle AIF$, $IA = IF \operatorname{cosec} IAF$. $\therefore$ ***IA = r cosec $\frac{1}{2}$A.***

Similarly, ***IB = r cosec $\frac{1}{2}$B*** and ***IC = r cosec $\frac{1}{2}$C.***

94. Ex-radii of a triangle.



Let I_1 be the centre and r_1 the radius of the escribed circle (opposite to the angle A) of the $\triangle ABC$; let D, E, F be the points of contact of the circle with the sides BC, and AC and AB produced.

Let r_2, r_3 denote the radii of the escribed circles opposite to the angles B and C respectively.

Now, $I_1D = I_1E = I_1F = r_1$; join AI_1, BI_1, CI_1.

$$\begin{aligned}\triangle ABC &= \triangle I_1AB + \triangle I_1AC - \triangle I_1BC \\ &= \tfrac{1}{2}AB.I_1F + \tfrac{1}{2}AC.I_1E - \tfrac{1}{2}BC.I_1D \\ &= \tfrac{1}{2}cr_1 + \tfrac{1}{2}br_1 - \tfrac{1}{2}ar_1 \\ &= \tfrac{1}{2}r_1(b+c-a) = \tfrac{1}{2}r_1(b+c+a-2a) \\ &= \tfrac{1}{2}r_1(2s-2a) \\ &= r_1(s-a).\end{aligned}$$

Thus, $\triangle = r_1(s-a)$.

$$\left.\begin{aligned} \therefore \quad \mathbf{r_1} &= \frac{\triangle}{\mathbf{s-a}} \\ \text{Similarly,} \quad \mathbf{r_2} &= \frac{\triangle}{\mathbf{s-b}} \\ \text{and} \quad \mathbf{r_3} &= \frac{\triangle}{\mathbf{s-c}} \end{aligned}\right\} \quad \cdots \text{(i)}$$

Again,
$$\begin{aligned} a &= BC = BD + CD \\ &= r_1 \cot I_1BD + r_1 \cot I_1CD, \\ &\qquad \text{from } \triangle^s\ I_1BD,\ I_1CD \\ &= r_1 \cot(90^\circ - \tfrac{1}{2}B) + r_1 \cot(90^\circ - \tfrac{1}{2}C),\end{aligned}$$

because, $\angle I_1BD = \tfrac{1}{2}(180^\circ - B) = 90^\circ - \tfrac{1}{2}B$,

and $\angle I_1CD = \tfrac{1}{2}(180^\circ - C) = 90^\circ - \tfrac{1}{2}C$.

$$\begin{aligned}\therefore \quad a &= r_1(\tan \tfrac{1}{2}B + \tan \tfrac{1}{2}C) \\ &= r_1\left[\frac{\sin \tfrac{1}{2}B}{\cos \tfrac{1}{2}B} + \frac{\sin \tfrac{1}{2}C}{\cos \tfrac{1}{2}C}\right] \\ &= r_1\left[\frac{\sin \tfrac{1}{2}B \cos \tfrac{1}{2}C + \sin \tfrac{1}{2}C \cos \tfrac{1}{2}B}{\cos \tfrac{1}{2}B \cos \tfrac{1}{2}C}\right] \\ &= r_1 \frac{\sin(\tfrac{1}{2}B + \tfrac{1}{2}C)}{\cos \tfrac{1}{2}B \cos \tfrac{1}{2}C} \\ &= r_1 \frac{\cos \tfrac{1}{2}A}{\cos \tfrac{1}{2}B \cos \tfrac{1}{2}C}, \text{ as in Art. 93.}\end{aligned}$$

$$\therefore \quad r_1 = a \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C \sec \tfrac{1}{2}A.$$

Putting $a = 2R \sin A = 4R \sin \frac{1}{2}A \cos \frac{1}{2}A$,

$$\left.\begin{aligned} \mathbf{r_1} &= \mathbf{4R \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.} \\ \text{Similarly, } \mathbf{r_2} &= \mathbf{4R \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B \cos \tfrac{1}{2}C,} \\ \text{and } \mathbf{r_3} &= \mathbf{4R \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \sin \tfrac{1}{2}C.} \end{aligned}\right\} \quad \cdots \text{(ii)}$$

Again, $AE = AC + CE = b + CD$ $[\because CE = CD]$

and $AF = AB + BF = c + BD$ $[\because BF = BD]$

But $AE = AF$; therefore, by addition, we get

$$2AE = b + c + BD + CD = b + c + a = 2s.$$

$\therefore \quad AE = s.$

Again, from ΔAI_1E, $I_1E = AE \tan I_1AE$.

$$\left.\begin{aligned} \therefore \quad \mathbf{r_1} &= \mathbf{s \tan \tfrac{1}{2}A.} \\ \text{Similarly, } \mathbf{r_2} &= \mathbf{s \tan \tfrac{1}{2}B,} \\ \text{and } \mathbf{r_3} &= \mathbf{s \tan \tfrac{1}{2}C.} \end{aligned}\right\} \quad \cdots \quad \cdots \text{(iii)}$$

Note. *Distances of Ex-centres from the vertices.*

From ΔAI_1F, $I_1A = I_1F \operatorname{cosec} I_1AF$.

$\therefore \quad \boldsymbol{I_1A = r_1 \operatorname{cosec} \tfrac{1}{2}A}$

$\quad\quad \boldsymbol{= 4R \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C}$ [*by formula (ii)*]

From ΔBI_1F, $I_1B = I_1F \operatorname{cosec} I_1BF$.

$\therefore \quad \boldsymbol{I_1B = r_1 \sec \tfrac{1}{2}B}$ $[\because \angle I_1BF = 90^\circ - \frac{1}{2}B]$

Similarly, $\boldsymbol{I_1C = r_1 \sec \tfrac{1}{2}C}$.

In the same way, $\boldsymbol{I_2B = r_2 \operatorname{cosec} \tfrac{1}{2}B}$, $\boldsymbol{I_3C = r_3 \operatorname{cosec} \tfrac{1}{2}C}$.

Ex. 1. *Prove that* $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r}$.

By formula (i), Art. 94,

$$\text{left side} = \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta}$$

$$= \frac{3s - (a+b+c)}{\Delta} = \frac{3s - 2s}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}.$$

Ex. 2. *Prove that* $4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C=\frac{s}{R}$.

$$\text{Left side}=4.\sqrt{\frac{s(s-a)}{bc}}.\sqrt{\frac{s(s-b)}{ca}}.\sqrt{\frac{s(s-c)}{ab}}$$

$$=\frac{4s}{abc}\cdot\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\frac{4s}{abc}\Delta=s\cdot\frac{4\Delta}{abc}=\frac{s}{R}\text{ by formula (ii), Art. 92.}$$

Ex. 3. *Show that*

$$\frac{bc-r_2r_3}{r_1}=\frac{ca-r_3r_1}{r_2}=\frac{ab-r_1r_2}{r_3}.$$

$$r_2r_3=\frac{\Delta^2}{(s-b)(s-c)}=s(s-a).$$

$$\therefore\quad bc-r_2r_3=\tfrac{1}{4}[4bc-2s(2s-2a)]$$
$$=\tfrac{1}{4}[4bc-(a+b+c)(b+c-a)]$$
$$=\tfrac{1}{4}[4bc+a^2-(b+c)^2]=\tfrac{1}{4}[a^2-(b-c)^2]$$
$$=\tfrac{1}{4}[(a+b-c)(a-b+c)]=(s-b)(s-c).$$

$$\therefore\quad\frac{bc-r_2r_3}{r_1}=\frac{(s-b)(s-c)}{r_1}=\frac{(s-a)(s-b)(s-c)}{\Delta}$$
$$=\frac{\Delta}{s}=r.$$

Similarly, the other ratios are equal to the same quantity.

Ex. 4. *Prove that in any triangle,*

$$r_1+r_2+r_3-r=4R.$$

$$\text{Left side}=\left(\frac{\Delta}{s-a}+\frac{\Delta}{s-b}\right)+\left(\frac{\Delta}{s-c}-\frac{\Delta}{s}\right)$$

$$=\Delta\cdot\frac{2s-(a+b)}{(s-a)(s-b)}+\Delta\cdot\frac{c}{s(s-c)}$$

$$=\Delta c\left[\frac{1}{(s-a)(s-b)}+\frac{1}{s(s-c)}\right]\quad[\because 2s=a+b+c.]$$

$$=\Delta c\left[\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right]$$

Now, Numerator $= 2s^2 - s(a+b+c) + ab$

$$= 2s^2 - s.2s + ab = ab.$$

Denominator $= \Delta^2$.

$\therefore$ Left side $= \dfrac{abc}{\Delta} = 4R$.

Ex. 5. *If $r_1 = r_2 + r_3 + r$, prove that the triangle is right-angled.*

From the given relation, we have

$$r_1 - r = r_2 + r_3,$$

or, $$\frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c},$$

or, $$\frac{\Delta.a}{s(s-a)} = \frac{\Delta(2s-b-c)}{(s-b)(s-c)} = \frac{\Delta.a}{(s-b)(s-c)}.$$

$\therefore \quad s(s-a) = (s-b)(s-c)$.

$\therefore \quad \tan^2 \frac{1}{2}A = \dfrac{(s-b)(s-c)}{s(s-a)} = 1. \quad \therefore \quad \tan \frac{1}{2}A = 1.$

$\therefore \quad \frac{1}{2}A = 45^\circ. \qquad \therefore \quad A = 90^\circ.$

Note. Although we get $\tan \frac{1}{2}A = \pm 1$, we reject the negative value because $\frac{1}{2}A$ is an acute angle.

Examples XIV(b)

Prove that in any triangle (*Ex. 1 to 14*) :—

1. $\sin A + \sin B + \sin C = \dfrac{s}{R}$.

2. $\cos A + \cos B + \cos C = 1 + \dfrac{r}{R}$.

[*Use* $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$.]

3. $\dfrac{b-c}{r_1} + \dfrac{c-a}{r_2} + \dfrac{a-b}{r_3} = 0$.

4. $r_2r_3+r_3r_1+r_1r_2=s^2$.

5. $r=R(\cos A+\cos B+\cos C-1)$.

6. $r_1=R(\cos B+\cos C-\cos A+1)$.

[*Use* $\cos B+\cos C-\cos A=-1+4\sin\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$]

7. $a\cos B\cos C+b\cos C\cos A+c\cos A\cos B=\dfrac{\Delta}{R}$.

8. $a\cot A+b\cot B+c\cot C=2(R+r)$.

$\left[a\cot A=\dfrac{a}{\sin A}\cdot\cos A=2R\cos A.\quad \textit{Then use Ex. 2.}\right]$

9. $R=\dfrac{1}{4}\,\dfrac{(r_2+r_3)(r_3+r_1)(r_1+r_2)}{r_2r_3+r_3r_1+r_1r_2}$.

10. $\Delta=\sqrt{rr_1r_2r_3}=r^2\cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C$.

11. $\left(\dfrac{1}{r}-\dfrac{1}{r_1}\right)\left(\dfrac{1}{r}-\dfrac{1}{r_2}\right)\left(\dfrac{1}{r}-\dfrac{1}{r_3}\right)=\dfrac{4R}{r^2s^2}=\dfrac{16R}{r^2(a+b+c)^2}$.

[*A. I. 1938*]

12. $\left(\dfrac{1}{r}+\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}\right)^2=\dfrac{4}{r}\left(\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}\right)$.

13. $r_1(r_2+r_3)\operatorname{cosec}A=r_2(r_3+r_1)\operatorname{cosec}B$
$$=r_3(r_1+r_2)\operatorname{cosec}C.$$

14. $\dfrac{bc}{r_1}+\dfrac{ca}{r_2}+\dfrac{ab}{r_3}=2R\left\{\dfrac{b}{a}+\dfrac{c}{a}+\dfrac{c}{b}+\dfrac{a}{b}+\dfrac{a}{c}+\dfrac{b}{c}-3\right\}$.

15. In a triangle, $a=13, b=14, c=15$; find r and R.

16. If a, b, c are in A.P., show that r_1, r_2, r_3 are in H.P.

17. If in a triangle, $3R=4r$, show that
$$4(\cos A+\cos B+\cos C)=7.$$

18. If the diameter of an ex-circle be equal to the perimeter of the triangle, show that the triangle is right-angled.

[*Use* $r_1=s\tan\frac{1}{2}A$.]

19. If $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right)=2$, show that the triangle must be right-angled.

20. If $8R^2=a^2+b^2+c^2$, show that the triangle is right-angled.

21. If S be the area of the in-circle and S_1, S_2, S_3 the areas of the escribed circles, then

$$\frac{1}{\sqrt{S}}=\frac{1}{\sqrt{S_1}}+\frac{1}{\sqrt{S_2}}+\frac{1}{\sqrt{S_3}}.$$

22. In any triangle, prove that the area of the in-circle is to the area of the triangle as π : cot $\frac{1}{2}A$ cot $\frac{1}{2}B$ cot $\frac{1}{2}C$.

23. If p_1, p_2, p_3 are the perpendiculars from the angular points of a triangle to the opposite sides, show that

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}.$$

24. If x, y, z be the lengths of the perpendiculars from the circum-centre on the sides BC, CA, AB of the triangle ABC, prove that

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{4xyz}.$$

25. If x, y, z are respectively equal to IA, IB, IC, and α, β, γ are respectively equal to I_1A, I_2B, I_3C, show that

(i) $\frac{xyz}{abc}=\frac{r}{s}$. (ii) $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$.

(iii) $\frac{bc}{\alpha^2}+\frac{ca}{\beta^2}+\frac{ab}{\gamma^2}=1$. (iv) $ax^2+by^2+cz^2=abc$.

[*Use Notes of Arts. 93 and 94.*]

CHAPTER XV

SOLUTION OF TRIANGLES

95. In a triangle, there are six parts, the three sides and the three angles. These are not independent, but are connected by the relations between the sides and angles of the triangle, which have been established in Chapter XIV. In fact, if three of the parts are given, the remaining three can, in general, be determined, and the corresponding triangle completely known. The cases that can arise are the following :

(1) three sides may be given

(2) three angles may be given

(3) two sides and the included angle may be given

(4) two angles and one side may be given

(5) two sides and an opposite angle may be given.

We shall discuss these cases one by one.

96. Three sides given.

Let the three sides a, b, c of a triangle ABC be given. Now, provided the sum of any two of these given sides is greater than the third, the triangle ABC with the three given sides can be geometrically constructed and the triangle is unique ; in other words, its angles are definite. To determine any angle A say, we may use the rigorous formula,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

and thereby determine cos A, and then from the cosine-table find out the angle with this cosine. It is clear that the angle, being an angle of a triangle, lies between 0 and π, and within this range an angle with a given cosine has got only one value. Thus the angle is definitely known.

Here we want to make one point clear. Though the formula used is rigorous, the Cosine-table, by means of which we determine the angle with a given cosine, gives only approximate values. Now, it is a principle proved in books on higher mathematics (with the aid of calculus), that *when an angle is determined by using an approximate table, the best result is obtained by using the Logarithmic tangent-table*, and an angle determined from its L tan, using a four-figure table is more accurate than that determined by using even a seven-figure sine-table or cosine-table. If a suitable tangent formula is available therefore, we should make use of it.

Hence, for practical purposes, in this case, to determine A, we use the formula

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

where $s = \frac{1}{2}(a+b+c)$, which is known.

Taking logarithm, and adding 10, we get the value of $L \tan \frac{1}{2}A$ and therefore A is known.

Similarly, B and C are determined.

In case $\tan \frac{1}{2}A$ happens to be equal to the tangent of a standard angle, $\frac{1}{2}A$ is at once known and the use of logarithm is not wanted.

Ex. *The sides of a triangle are* 2, 3, 4. *Find the greatest angle, having given*

$$\log 2 = \cdot 30103, \qquad \log 3 = \cdot 4771213,$$

$$L \tan 52^\circ\ 14' = 10\cdot 1108395, \quad L \tan 52^\circ\ 15' = 10\cdot 1111004.$$

Here, $\quad s = \dfrac{2+3+4}{2} = \dfrac{9}{2}.$

The greatest side 4 being denoted by 'a', the greatest angle A (which is opposite to the greatest side) is given by

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(\frac{9}{2}-2)(\frac{9}{2}-3)}{\frac{9}{2}(\frac{9}{2}-4)}} = \sqrt{\frac{5.3}{9.1}} = \sqrt{\frac{10}{2.3}}.$$

$$\therefore \quad L \tan \tfrac{1}{2}A = 10 + \tfrac{1}{2}(\log 10 - \log 2 - \log 3)$$

$$= 10 + \tfrac{1}{2}(1 - \cdot 30103 - \cdot 4771213)$$

$$= 10\cdot 1109244.$$

Now, $L \tan \frac{1}{2}A$ lies between $L \tan 52^\circ 14'$ and $L \tan 52^\circ 15'$.

Hence, $\frac{1}{2}A$ lies between $52^\circ\ 14'$ and $52^\circ\ 15'$.

Let $\quad \frac{1}{2}A = 52^\circ\ 14'\ x''$.

Then diff. for x'' is $\cdot 0000849$,

and diff. for $1'$ *i.e.*, $60''$ is $\cdot 0002609$.

Hence, $\dfrac{x}{60} = \dfrac{849}{2609}$, or, $x = \dfrac{60 \times 849}{2609} = 19\cdot 5$ nearly.

Hence, $\frac{1}{2}A = 52^\circ\ 14'\ 19''\cdot 5$,

or, $\quad A = 104^\circ\ 28'\ 39''$ nearly.

97. Three angles given.

In this case the triangle cannot be solved, for there are innumerable triangles with the same three angles. All these

triangles, being equiangular, are similar, and the ratio of their sides can be determined from the formula,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C},$$

or, $a : b : c = \sin A : \sin B : \sin C$.

Ex. *The angles of a triangle are in the ratio* $2 : 3 : 7$. *Prove that the sides are in the ratio of* $\sqrt{2} : 2 : (\sqrt{3}+1)$.

The angles being in the ratio of $2 : 3 : 7$, and their sum being $180°$, the angles are evidently $30°$, $45°$ and $105°$ respectively. Hence, the ratio of the sides will be

$$\sin 30° : \sin 45° : \sin 105°,$$

i.e., $$\frac{1}{2} : \frac{1}{\sqrt{2}} : \frac{\sqrt{3}+1}{2\sqrt{2}},$$

or, $$\sqrt{2} : 2 : (\sqrt{3}+1).$$

Examples XV(a)

1. The sides of a triangle are 24, 22, 14 ; find the least angle, given $L \tan 17° 33' = 9\text{·}500042$, diff. for $1' = 439$.

2. The sides of a triangle are 50, 36 and 28 ; find the greatest angle, having given

$$\log 19 = 1\text{·}2787536, \qquad \log 29 = 1\text{·}4623980$$

$$L \tan 51° 0' = 10\text{·}0916308, \; L \tan 51° 1' = 10\text{·}0918891.$$

3. The sides of a triangle are 9, 10 and 11 ; find the angle opposite to the side 10, given

$$L \tan 29° 30' = 9\text{·}7526420, \; L \tan 29° 29' = 9\text{·}7523472$$

$$\log 2 = \text{·}30103.$$ [*C. U. 1943*]

4. The sides of a triangle are 2, 3, 4. Find all the angles correctly to degrees and minutes by the help of mathematical tables.

5. (i) The sides of a triangle are 15, 19, 24 ; find the greatest angle of the triangle.

Given $\log 5{\cdot}7 = {\cdot}75587$, $L \cos 88^\circ\ 59' = 8{\cdot}24903$
diff. for $1' = 718$. [*C. U. 1936*]

(ii) Find the greatest angle in degrees, minutes and seconds in a triangle whose sides are 5, 6, 7, having given

$\log 6 = {\cdot}7781513$

$L \cos 39^\circ\ 14' = 9{\cdot}8890644$, diff. for $60'' = {\cdot}0001032$.

6. (i) The sides of a triangle are 7, 8, 9 ; solve the triangle. [*C. U. 1938*]

(ii) If $a = 35$, $b = 40$, $c = 66$, determine the greatest angle. [*C. U. 1945*]

[Use *Mathematical Tables*]

7. Given $a = \sqrt{6}$, $b = 2$, $c = \sqrt{3} - 1$; solve the triangle.

8. Given $a = 2$, $b = \sqrt{2}$, $c = \sqrt{3} + 1$; solve the triangle.

9. If $a = 7$, $b = 5$, $c = 8$, solve the triangle.
Given $\cos 38^\circ\ 11' = \frac{11}{14}$.

10. If $a = 3 + \sqrt{3}$, $b = 2\sqrt{3}$, $c = \sqrt{3}$, solve the triangle.

11. The angles of a triangle are 105°, 60° and 15° ; find the ratio of the sides.

12. If $A = 45^\circ$, $B = 60^\circ$, show that $c : a = \sqrt{3} + 1 : 2$.

13. The angles of a triangle are as $1 : 2 : 7$; find the ratio of the greatest side to the least side.

14. If $\cos A = \frac{4}{5}$, $\cos B = \frac{3}{5}$, find $a : b : c$.

15. If the angles adjacent to the base of a triangle are $22\frac{1}{2}^\circ$ and $112\frac{1}{2}^\circ$, show that the altitude is half the base.

16. If the sides of a triangle are 4, 5, 6, show that the greatest angle is double the least.

98. Two sides and the included angle given.

Let the two sides b, c and the included angle A of a triangle ABC be given. It is easy to construct the triangle geometrically, and there will be only one definite triangle with the given parts. To find the other angles B and C, we notice that

$$B+C=180^\circ-A,$$

$$i.e., \quad \frac{B+C}{2}=90^\circ-\frac{A}{2}.$$

Again,

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}.$$

$$\therefore \quad L\tan\frac{B-C}{2}=10+\log\left(\frac{b-c}{b+c}\cot\frac{A}{2}\right)$$

$$=\log\left(\frac{b-c}{b+c}\right)+L\cot\frac{A}{2}.$$

b, c, and A being given, the right-hand side is known and thus, $L\tan\frac{B-C}{2}$ is known, whence $\frac{B-C}{2}$ is known.

Now $\frac{B+C}{2}$ and $\frac{B-C}{2}$ being both known, by addition and subtraction, we get B and C respectively.

The reason of using tangent formula to determine $\frac{B-C}{2}$ is already explained in Art. 96.

When B and C are known, the third side a is easily obtained from

$$\frac{a}{\sin A}=\frac{b}{\sin B}, \text{ or, } =\frac{c}{\sin C}.$$

Ex. *In a triangle,* $b=2{\cdot}25$, $c=1{\cdot}75$, $A=54^\circ$, *find* B *and* C, *having given,*

$$\log 2={\cdot}301030, \qquad L\tan 63^\circ=10{\cdot}292834$$

$$L\tan 13^\circ\,47'=9{\cdot}389724, \quad L\tan 13^\circ\,48'=9{\cdot}390270.$$

[*C. U. 1931*]

Here,

$$\frac{B+C}{2}=90^\circ-\frac{A}{2}=90^\circ-27^\circ=63^\circ. \quad \dots (i)$$

Again,

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{\cdot 5}{4}\cot 27^\circ$$

$$=\tfrac{1}{8}\tan 63^\circ.$$

$$\therefore \quad L\tan\frac{B-C}{2}=L\tan 63^\circ-3\log 2$$

$$=10{\cdot}292834-{\cdot}903090$$

$$=9{\cdot}389744.$$

Now, $L\tan 13^\circ 47'=9{\cdot}389724$

and $L\tan 13^\circ 48'=9{\cdot}390270.$

Hence, $\dfrac{B-C}{2}$ being $13^\circ\ 47'\ x''$

we get, diff. for $x''={\cdot}000020$

and diff. for $1'$ *i.e.*, $60''={\cdot}000546.$

$$\therefore \quad \frac{x}{60}=\frac{20}{546}, \quad \text{or,} \quad x=\frac{20\times 60}{546}=2{\cdot}2 \text{ nearly.}$$

Hence, $\dfrac{B-C}{2}=13^\circ\ 47'\ 2''{\cdot}2$ nearly.

Combining with (i), $B=76^\circ\ 47'\ 2''{\cdot}2$ and $C=49^\circ\ 12'\ 57''{\cdot}8$.

99. Two angles and a side given.

Let any side a of a triangle ABC, and any two of its angles be given. The sum of the three angles being 180°, the third angle is also known. To find the other two sides b and c, we use the formula

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

Ex. *In a triangle ABC, $A=38^\circ 20'$, $B=45^\circ$ and $b=64$ ft. Find c, having given $\log 2={\cdot}30103$, $L\sin 83^\circ\ 20'=9{\cdot}99705$ and $\log{\cdot}089896=\bar{2}{\cdot}95374$.*

Here,

$$C = 180^\circ - (A + B)$$
$$= 180^\circ - 83^\circ\ 20'.$$

Now,

$$\frac{c}{\sin C} = \frac{b}{\sin B},$$

or, $$\frac{c}{\sin(180^\circ - 83^\circ\ 20')} = \frac{64}{\sin 45^\circ} = \frac{64}{1/\sqrt{2}} = 64\sqrt{2}.$$

$\therefore \quad c = 2^{\frac{13}{2}} \sin 83^\circ\ 20'$;

$\therefore \quad \log c = \frac{13}{2} \log 2 + L \sin 83^\circ\ 20' - 10$
$$= \tfrac{13}{2}(\cdot 30103) + 9 \cdot 99705 - 10 = 1 \cdot 95374.$$

Thus, log c has the same mantissa as log ·089896, but has its characteristic 1. Hence, $c = 89 \cdot 896$ feet.

Examples XV(b)

1. Two sides of a triangle are 3 and 5 feet and the included angle is 120°; find the other angles, having given

log 4·8 = ·6812412

L tan $8^\circ\ 12' = 9 \cdot 1586706$, diff. for $60'' = 8940$.

[*C. U. 1949*]

2. If $b = 1300$, $c = 1400$ and $A = 60^\circ$, find B and C.

Given log 3 = ·4771213,

L tan $3^\circ\ 40' = 8 \cdot 8067422$, diff. for $10'' = 3306$.

3. If $a = 21$, $b = 11$, $C = 34^\circ\ 42'\ 30''$, find A and B.

Given log 2 = ·30103,

and L tan $72^\circ\ 38'\ 45'' = 10 \cdot 50515$.

4. If the sides a and b are in the ratio 7 : 3 and the included angle C is 60°, find A and B, given

log 2 = ·3010300, log 3 = ·4771213

L tan $34^\circ\ 42' = 9 \cdot 8403776$, diff. for $1' = 2699$.

5. Two sides of a plane triangle are 14 and 11 and the included angle is 60°. Find the remaining angles, having given $L \tan 11^\circ 41' = 9{\cdot}3174299$, $L \tan 11^\circ 45' = 9{\cdot}3180640$.
[*C. U. 1932*]

6. (i) Two sides of a triangle are 80 and 100 ft. and the included angle is 60°. Find the other angles. [*C. U. 1946*]

(ii) If $a = 5$, $b = 3$, $C = 70^\circ 30'$, find the remaining angles.

(iii) If $a = 39{\cdot}9$, $b = 43{\cdot}2$, $C = 38^\circ 14'$, solve the triangle.
[Use *Mathematical Tables*]

7. (i) In a plane triangle, $b = 540$, $c = 420$ and $A = 52^\circ 6'$; find B and C, having given

$L \tan 26^\circ 3' = 9{\cdot}6891430$,

$L \tan 14^\circ 20' = 9{\cdot}4074189$, $L \tan 14^\circ 21' = 9{\cdot}4079153$.
[*C. U. 1934*]

(ii) Given $a = 70$, $b = 35$, $C = 36^\circ 52' 12''$, $\log 3 = 0{\cdot}4771213$, $L \cot 18^\circ 26' 6'' = 10{\cdot}4771213$. Calculate the other two angles A and B. [*C. U. 1935, '37*]

8. If $a = 2\sqrt{6}$, $c = 6 - 2\sqrt{3}$, $B = 75^\circ$, solve the triangle.

9. Two sides of a triangle are $\sqrt{3} + 1$ and $\sqrt{3} - 1$ and the included angle is 60°; solve the triangle.

10. (i) If $a = 2$, $b = 1 + \sqrt{3}$, $C = 60^\circ$, solve the triangle.

(ii) If $a = 2$, $b = 4$, $C = 60^\circ$, find A and B.

11. If $a = 19$, $B = 52^\circ 28'$ and $C = 93^\circ 40'$, find b, having given $\log 27038 = 4{\cdot}4319746$; $\log 19 = 1{\cdot}2787536$;
$\log 27037 = 4{\cdot}4319585$;
$L \sin 52^\circ 28' = 9{\cdot}8992727$, $L \sin 33^\circ 52' = 9{\cdot}7460595$.

12. If $B = 45^\circ$, $C = 10^\circ$ and $a = 200$ ft., find b, having given
$\log 2 = {\cdot}30103$, $L \sin 55^\circ = 9{\cdot}9133645$
$\log 1726{\cdot}4 = 3{\cdot}2371414$, $\log 1726{\cdot}5 = 3{\cdot}2371666$.
[*C. U. 1947*]

13. If $A = 41^\circ\ 13'\ 22''$, $B = 71^\circ\ 19'\ 5''$, and $a = 55$, find b, given $\log 55 = 1.7403627$, $\log 79063 = 4.8979775$

$L \sin 41^\circ\ 13'\ 22'' = 9.8188779$

$L \sin 71^\circ\ 19'\ 5'' = 9.9764927.$

14. (i) If $B = 70^\circ\ 30'$, $C = 78^\circ\ 10'$, $a = 102$, solve the triangle.

(ii) If $a = 39$, $A = 81^\circ 35'$, $B = 27^\circ 55'$, solve the triangle.

(iii) If $A = 37^\circ\ 15'$, $B = 72^\circ\ 5'$, $a = 75.2$, find b and c.

[*Mathematical tables should be used*]

15. If $A = 75^\circ$, $B = 30^\circ$, $b = \sqrt{8}$, solve the triangle.

16. If $A = 30^\circ$, $B = 45^\circ$, $b = 2$, solve the triangle.

17. In a triangle in which each base angle is double of the third angle, the base 2 ; solve the triangle.

18. Given $a = \sqrt{57}$, $A = 60^\circ$, $\Delta = 2\sqrt{3}$, find b and c.

100. Two sides and an opposite angle given.

Let the two sides b and c in a triangle ABC, and the angle B opposite to the side b be given.

In this case, we get the angle C from the formula,

$$\frac{\sin C}{c} = \frac{\sin B}{b}, \text{ or, } \sin C = \frac{c \sin B}{b}.$$

Now, three cases may arise, namely,

(i) $c \sin B > b$. In this case $\sin C$ is greater then 1, and so C cannot be obtained. In fact in this case no triangle is possible.

(ii) $c \sin B = b$. Here, $\sin C$ becomes 1 and therefore, $C = 90^\circ$. Thus, $A = 90^\circ - B$. We thus get a right-angled triangle with right angle at C, and the side a will be obtained from

$$c^2 = a^2 + b^2, \text{ or, } a = \sqrt{c^2 - b^2}.$$

(iii) $c \sin B < b$. In this case $\sin C$ is less than 1, and hence, C can be determined. Now, sines of supplementary angles are known to be equal, and an angle of a triangle may be acute or obtuse. We therefore get two supplementary values of C having the same value for $\sin C$. Three sub-cases now arise :

Sub-case A. If of the two given sides, $b > c$, then $B > C$, and therefore the obtuse value of C becomes inadmissible, for otherwise B is also obtuse and two angles B and C of the triangle become both obtuse. Thus, the only admissible solution is the acute value of C. Now, B and C being both known, A is obtained from $A+B+C=180°$. The side a will be known from

$$\frac{a}{\sin A}=\frac{b}{\sin B}, \text{ or, } \frac{c}{\sin C}.$$

Thus, the triangle is uniquely solved.

Sub-case B. If $b=c$, then $B=C$, and here also the obtuse value of C is admissible ; with the acute value of C the triangle is uniquely solved exactly as in the above case.

Sub-case C. If $b < c$, then $B < C$, so that C may be either acute or obtuse. Both the supplementary values of C being admissible now, there will be two possible triangles with the three given parts b, c, B. Corresponding to each value of C, the value of A is determined from $A+B+C=180°$, and then a is obtained from the formula,

$$\frac{a}{\sin A}=\frac{b}{\sin B}, \text{ or, } \frac{c}{\sin C}.$$

As there are two solutions of the triangle in this case, each equally admissible, this sub-case in the solution of a triangle in which b, c, B are given and $b > c \sin B$ but $< c$, is referred to as the **Ambiguous Case** in the solution of triangles.

We may sum up the results as follows :

When in a triangle, b, c, B are given,

(i) if $b < c \sin B$, no triangle is possible ;

(ii) if $b = c \sin B$, we get a definite right-angled triangle as solution ;

(iii) if $b > c$ and therefore necessarily $> c \sin B$, we get one definite solution having C acute ;

(iv) if $b = c$ and therefore necessarily $> c \sin B$, we get one definite solution having C acute.

(v) if $b > c \sin B$ but $< c$, there are two solutions, and this case is the *ambiguous case*.

101. Geometrical treatment of the Ambiguous Case.

To make the ideas clear, we proceed to construct geometrically the triangle in which two sides and an opposite angle, *viz.*, b, c and B are given.

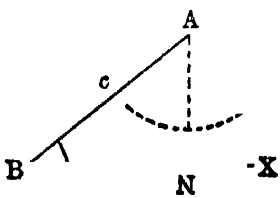


Fig. (i)

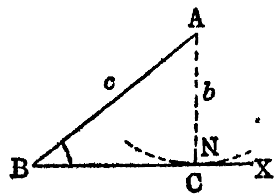


Fig. (ii)

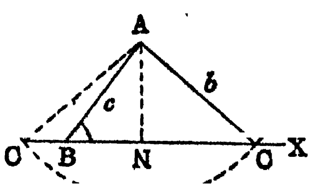


Fig. (iii)

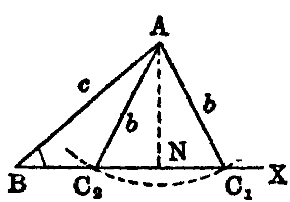


Fig. (iv)

Let ABX be the given angle B, and along one arm of it, take $AB = c$. Let AN be the perpendicular from A on BX. Then $\frac{AN}{AB} = \sin B$, so that $AN = AB \sin B = c \sin B$.

With centre A and radius b draw a circle.

Case (*i*). If $b < c \sin B$, *i.e.*, $< AN$, the circle does not meet the side BX at all and no triangle is therefore obtained. [See *fig*. (*i*)]

Case (*ii*). If $b = c \sin B$, *i.e.*, $= AN$, the circle touches the side BX at C coincident with N, as in fig. (ii). Hence, a right-angled triangle is formed, in which the sides AB, AC and the angle B have the given values c, b, B. Thus, ABC is the required triangle.

Case (*iii*). If $b > c$, *i.e.*, $> AB$, the circle cuts BX at two points C and C' on opposite sides of B as in fig. (iii). The triangle ABC', though it has the sides AB, AC equal to the given quantities c and b, has the angle B not equal to the given angle, but equal to its supplement. Hence it is not the solution required. In this case the triangle ABC is the only solution.

Case (*iv*). If $b = c$, *i.e.*, $= AB$, the point C' of the above case coincides with B, and only one triangle ABC is obtained as the required solution.

Case (*v*). If $b > c \sin B$, *i.e.*, $> AN$ but less than c (or, AB), the circle cuts BX at two points C_1 and C_2 on the same side of B as in fig. (iv). Both the triangles ABC_1 and ABC_2 have the same three given parts and both are possible solutions. This is therefore the *Ambiguous* case.

Note. By considering the equation

$$b^2 = c^2 + a^2 - 2ac \cos B$$

in which b, c, B are given, we may first of all determine a, instead of trying to determine C.

Considering the equation as a quadratic in a, *viz.*,

$$a^2-2c\cos B.a+c^2-b^2=0,$$

and by solving it, we get

$$a=c\cos B\pm\sqrt{b^2-c^2\sin^2 B}.$$

(i) If $b<c\sin B$, $b^2-c^2\sin^2 B$ is negative and thus the two values of a are imaginary. (*No solution*)

(ii) If $b=c\sin B$, $b^2-c^2\sin^2 B=0$ and thus the two values of a are real and coincident.

(*one solution*: one triangle right-angled at C, since $b=c\sin B$)

(iii) If $b>c\sin B$, $b^2-c^2\sin^2 B$ is positive, so two values of a are real and distinct, but they are not always admissible.

(*a*) When $b>c$, $\{i.e., b^2>c^2(\sin^2 B+\cos^2 B)\}$, $b^2-c^2\sin^2 B>c^2\cos^2 B$, *i.e.*, $\sqrt{b^2-c^2\sin^2 B}>c\cos B$ and hence one value of a is positive and the other negative. (*one solution*)

(*b*) When $b=c$, $b^2-c^2\sin^2 B=c^2-c^2\sin^2 B=c^2\cos^2 B$ and hence one value of a is zero. (*one solution*)

(*c*) When $b<c$, *i.e.*, $b^2<c^2(\sin^2 B+\cos^2 B)$, $b^2-c^2\sin^2 B<c^2\cos^2 B$, *i.e.*, $\sqrt{b^2-c^2\sin^2 B}<c\cos B$.

So both the values of a are real and positive. (*two solutions*)

This is known as the *algebraical discussion* of the ambiguous case.

An example illustrating the algebraic method is added below.

Ex. 1. *In a triangle,* $b=15$ ft., $c=10$ ft., $B=60^\circ$. *Find* a *and* A *having given* $\sin 84^\circ 44'=\cdot 99578$.

We have $b^2=c^2+a^2-2ca\cos B$, giving here

$$225=100+a^2-20a\cos 60^\circ;$$

or, $a^2-10a-125=0$ whence

$$a=5\pm 5\sqrt{6}.$$

Rejecting the negative value for a as inadmissible, the only possible value of $a = 5(\sqrt{6}+1)$ ft. There is thus one solution and there is no ambiguity. In fact this is case (iii) of the previous article.

Again, $\sin A = \frac{a}{b} \sin B = \frac{5(\sqrt{6}+1)}{15} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{2}+\sqrt{3}}{6}$

$$= \frac{3 \times 1\cdot41421\cdots + 1\cdot73205}{6} = \cdot99578\cdots$$

so $A = 84° \ 44'$.

Ex. 2. *In a triangle, $a = 73\cdot4$, $b = 64\cdot9$ and $B = 48°13'25''$; find A, having given*

$log\ 734 = 2\cdot8656961, \qquad log\ 649 = 2\cdot8122447$

$L\ sin\ 48°\ 13'\ 25'' = 9\cdot8725936$

$L\ sin\ 57°\ 30' = 9\cdot9260292 \qquad (\textit{diff. for } 1' = 804)$

Is the case ambiguous?

Here,

$$\sin A = \frac{a \sin B}{b} = \frac{734}{649} \sin 48°\ 13'\ 25''$$

$$\therefore \quad L \sin A = \log 734 - \log 649 + L \sin 48°\ 13'\ 25''.$$
$$= 2\cdot8656961 - 2\cdot8122447 + 9\cdot8725936$$
$$= 9\cdot9260450.$$

Now, diff. of this from $L \sin 57°\ 30' = 158$ (*i.e.*, $\cdot0000158$)
and diff. for $1'$ (or $60''$) $= 804$ (*i.e.*, $\cdot0000804$).

Hence, $A = 57°\ 30'\ x''$ where

$$\frac{x}{60} = \frac{158}{804} \text{ whence } x = 11\cdot8 \text{ nearly.}$$

Thus, $A = 57°\ 30'\ 11\cdot8''$ or its supplement $122°\ 29'\ 48\cdot2''$ which has also the same sine, and so the same L sine.

Now, in this case $a > b$ and so $A > B$ and thus both values of A are admissible. The case is, therefore, the *ambiguous case* and will have two solutions.

Examples XV(c)

1. Given (i) $A=30^\circ$, $a=6$, $b=4$.
(ii) $A=60^\circ$, $a=7$, $b=8$.
(iii) $A=45^\circ$, $a=2$, $b=8$.
(iv) $A=30^\circ$, $a=3$, $b=6$.

Find in which case the solution is ambiguous, in which case there is one solution, and in which case there is nc solution.

2. (i) If $b=2$, $c=\sqrt{3}+1$ and $B=45^\circ$, solve the triangle.
(ii) If $a=3$, $b=3\sqrt{3}$, $A=30^\circ$, find B.

3. If $a=2$, $b=\sqrt{6}$, $B=60^\circ$, solve the triangle.

4. If $a=2$, $b=5$, $A=30^\circ$, solve the triangle.

5. If b, c, B are given and if $b<c$, show that

$$(a_1-a_2)^2+(a_1+a_2)^2\tan^2 B=4b^2$$

a_1 and a_2 being the two possible values of a.

6. In the ambiguous case, given a, b and A, prove that the difference between the two values of c is

$$2\sqrt{a^2-b^2\sin^2 A}.$$

7. If a, b, A are given, and if c_1, c_2 are the values of the third side in the ambiguous case, prove that if $c_1>c_2$,

(i) $c_1-c_2=2a\cos B_1$. [*B. H. U. I. 1928*]

(ii) $c_1^2+c_2^2-2c_1c_2\cos 2A=4a^2\cos^2 A$.
[*B. H. U. I. 1935* ; *Pat. I. 1936*]

(iii) $\cos\dfrac{C_1-C_2}{2}=\dfrac{b\sin A}{a}$. [*A. I. 1941*]

8. If $b=16$, $c=25$ and $B=33^\circ\,15'$, find the other angles ; given

$L\sin 33^\circ\,15'=9{\cdot}7390129$, $\log 2={\cdot}30103$,
$L\sin 58^\circ\,57'=9{\cdot}9328376$, $L\sin 58^\circ\,56'=9{\cdot}9327616$.

9. If $a=5$, $b=4$, $A=45^\circ$, find B and C; given

$$\log 2 = \cdot 30103, \; L \sin 31^\circ\, 27' = 9\cdot 75257.$$

10. If $a=30$, $b=300$, find A in order that B may be a right angle, having given that

$$L \sin 5^\circ\, 44' = 8\cdot 9995595, \text{ diff. for } 1' = 12565.$$

11. If $a=16$, $c=25$ and $C=60^\circ$, find the other angles; given

$$\log 2 = \cdot 30103, \qquad \log 3 = \cdot 4771213$$

$$L \sin 33^\circ\, 39' = 9\cdot 7436024, \text{ diff. for } 1' = 1897.$$

12. If $b=165$, $c=258$, and $B=35^\circ\, 10'$, find the angles A and C; given

$$\log 1\cdot 65 = \cdot 21749, \qquad \log 2\cdot 58 = \cdot 41162$$

$$L \sin 35^\circ\, 10' = 9\cdot 76039, \quad L \sin 64^\circ\, 14' = 9\cdot 95452.$$

13. If $2b=3a$ and $\tan^2 A = \frac{3}{5}$, prove that there are two values of the third side, one of which is double the other.

14. If A_1, B_1 and A_2, B_2 are the angles of the two triangles in the ambiguous case where b c, C are given,

$$\text{then } \frac{\sin A_1}{\sin B_1} + \frac{\sin A_2}{\sin B_2} = 2 \cos C.$$

15. Show that in the case that admits of two solutions, the two values of C satisfy the equation

$$\frac{(a+b)^2}{1+\cos C} + \frac{(b-a)^2}{1-\cos C} = \frac{2a^2}{\sin^2 A}. \quad \text{[B. H. U. I. 1942]}$$

16. If $\log b + 10 = \log c + L \sin B$, can the triangle be ambiguous?

Miscellaneous Examples II

In any triangle ABC, prove that (*Ex. 1 to 8*) :—

1. $\frac{1}{a}\cos A + \frac{1}{b}\cos B + \frac{1}{c}\cos C = \frac{a^2+b^2+c^2}{2abc}$.

2. $(b^2+c^2-a^2)\tan A = (c^2+a^2-b^2)\tan B$
$= (a^2+b^2-c^2)\tan C$.

3. $b^2+c^2-2bc\cos(A+60°) = c^2+a^2-2ca\cos(B+60°)$
$= a^2+b^2-2ab\cos(C+60°)$.

4. $(\cot\frac{1}{2}A - \tan\frac{1}{2}B - \tan\frac{1}{2}C)^{\frac{1}{2}}$
$+ (\cot\frac{1}{2}B - \tan\frac{1}{2}C - \tan\frac{1}{2}A)^{\frac{1}{2}} + (\cot\frac{1}{2}C - \tan\frac{1}{2}A - \tan\frac{1}{2}B)^{\frac{1}{2}}$
$= (\cot\frac{1}{2}A + \cot\frac{1}{2}B + \cot\frac{1}{2}C)^{\frac{1}{2}}$.

5. $a\sin(B-C)\cos(B+C-A) + b\sin(C-A)$
$\times\cos(C+A-B) + c\sin(A-B)\cos(A+B-C) = 0$.

6. $\frac{a\sin A + b\sin B + c\sin C}{a\cos A + b\cos B + c\cos C} = \frac{R}{abc}(a^2+b^2+c^2)$.

7. $(b+c-2a)\sin\frac{1}{2}A\sin\frac{1}{2}(B-C)$
$+ (c+a-2b)\sin\frac{1}{2}B\sin\frac{1}{2}(C-A)$
$+ (a+b-2c)\sin\frac{1}{2}C\sin\frac{1}{2}(A-B) = 0$.

8. $a\cos A\cos 2A + b\cos B\cos 2B + c\cos C\cos 2C$
$+ 4\cos A\cos B\cos C(a\cos A + b\cos B + c\cos C) = 0$.

9. If in a triangle, a^2, b^2, c^2 are in A.P., show that
$\tan A$, $\tan B$, $\tan C$ are in H.P.

10. If in a triangle, $\sin A$, $\sin B$, $\sin C$ are in H.P., show that $1-\cos A$, $1-\cos B$, $1-\cos C$ are in H.P.

11. Determine the triangle whose sides are three consecutive terms in the series of natural numbers and whose largest angle is double the least.

12. If in a triangle, $\cos 3A + \cos 3B + \cos 3C = 1$, show that one angle must be 120°.

13. If in a triangle, $\sin A$, $\sin B$, $\sin C$ be in A. P., show that $\tan \frac{1}{2}A \tan \frac{1}{2}C = \frac{1}{3}$.

14. If $a = 5$, $b = 7$ and $A = 30^\circ$, find B in degrees and minutes, having given

$$\sin 44^\circ = 0{\cdot}6947, \ \sin 45^\circ = 0{\cdot}7071.$$

15. In the *ambiguous case*, the area of one of the triangles is n times that of the other, show that if b be the greater of the given sides and c the less, $\frac{b}{c}$ is less than $\frac{n+1}{n-1}$.

16. In the *ambiguous case*, show that the circum-circles of the two triangles are equal.

17. Prove that

(i) $\tan^{-1}\left(\frac{x \cos \phi}{1 - x \sin \phi}\right) - \tan^{-1}\left(\frac{x - \sin \phi}{\cos \phi}\right) = \phi.$

(ii) $\tan^{-1}\frac{t_1 - t_2}{1 + t_1 t_2} + \tan^{-1}\frac{t_2 - t_3}{1 + t_2 t_3} + \cdots + \tan^{-1}\frac{t_{n-1} - t_n}{1 + t_{n-1} t_n}$

$$= \tan^{-1} t_1 - \tan^{-1} t_n.$$

18. If the sum of four angles be 180°, prove that the sum of the products of their cosines taken two and two together is equal to the sum of the products of their sines taken similarly.

19. Prove that $\cos^2 A + \cos^2\left(A + \frac{\pi}{3}\right) + \cos^2\left(A - \frac{\pi}{3}\right) = \frac{3}{2}$.

[*C. U. 1933*]

20. In a triangle ABC, if $\tan \frac{A}{2}$, $\tan \frac{B}{2}$, $\tan \frac{C}{2}$ be in Arithmetical progression then $\cos A$, $\cos B$, $\cos C$ are also in Arithmetical progression.

21. Give in general terms the solutions of the following equation :

$$\tan(x+b)\tan(x+c)+\tan(x+c)\tan(x+a)$$
$$+\tan(x+a)\tan(x+b)=1.$$

22. If $A+B+C=180°$, prove that

$$\left(1+\tan\frac{A}{4}\right)\left(1+\tan\frac{B}{4}\right)\left(1+\tan\frac{C}{4}\right)$$
$$=2\left(1+\tan\frac{A}{4}\tan\frac{B}{4}\tan\frac{C}{4}\right).$$

23. Prove that

$$\sin^2 x+\sin^2 y+\sin^2 z+\sin^2(x+y+z)$$
$$=2-2\cos(x+y)\cos(y+z)\cos(z+x).$$

24. Solve the following equation :

$$\tan x+\tan\left(x+\frac{\pi}{3}\right)+\tan\left(x+\frac{2\pi}{3}\right)=3.$$

[*Left side reduces to* $3\tan 3x$.]

25. Prove that in a triangle ABC,

$$\Delta=\frac{(a+b+c)^2}{4\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)}$$

26. Prove that

$$\log\sin 8x=3\log 2+\log\sin x+\log\cos x$$
$$+\log\cos 2x+\log\cos 4x.$$

27. Show that in any triangle ABC,

$$\log\tan\frac{A}{2}=\tfrac{1}{2}[\log(s-b)+\log(s-c)-\log s-\log(s-a)].$$

28. Prove that (i) $x^{\log y}=y^{\log x}$.

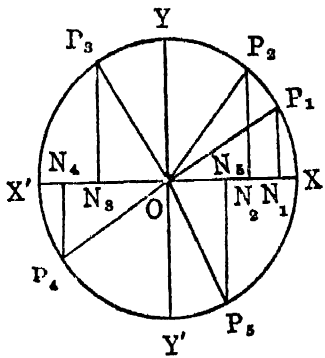
(ii) $x^{\log y-\log z}\times y^{\log z-\log x}\times z^{\log x-\log y}=1.$

29. In any right-angled triangle ABC, C being the right-angle show that $R+r=\frac{1}{2}(a+b)$.

30. Show how to solve a triangle having given the three perpendiculars from the vertices on the opposite sides.

GRAPHS OF TRIGONOMETRICAL FUNCTIONS

102. Changes in the Trigonometrical ratios of an angle *as the angle increases from* 0° to 360°.



Suppose an angle traced out by a revolving line starting from OX, changes gradually from 0° to 360°.

Take a circle with centre O of any radius. It is clear that in determining the trigonometrical ratios of an angle XOP_1 in its different positions, we can keep the hypotenuse OP_1 always the same, equal to the radius of the circle.

(i) Changes in sine.

When the angle N_1OP_1 ($=\theta$ say) is zero, its sine is zero. As the angle increases from 0° to 90°, the hypotenuse OP_1 remaining the same, the opposite side P_1N_1 is positive and gradually increases, as is evident by comparing the triangles N_1OP_1 and N_2OP_2.

Hence, $\sin\theta = \dfrac{P_1N_1}{OP_1}$ gradually increases, until when $\theta = 90^\circ$, P_2N_2 and OP_2 both coincide with OY and $\sin\theta$ attains its greatest value 1.

As θ still further increases, from 90° to 180°, the hypotenuse OP_3 retains the same value, but P_3N_3 remaining positive, now gradually diminishes from OY to zero, and so $\sin\theta$ diminishes from 1 to 0. In the third quadrant, as θ increases from 180° to 270°, P_4N_4 is negative and numerically increases from zero to OY', the hypotenuse remaining always positive and unaltered. $\sin\theta$ is therefore negative and numerically increases from 0 to 1; in other words, it diminishes gradually from 0 to -1. In the fourth quadrant, as θ increases from 270° to 360°, P_5N_5 remaining negative numerically diminishes from OY' to 0, and $\sin\theta$ therefore remaining negative numerically diminishes from 1 to 0; in other words, it increases from -1 to 0. The results are therefore as follows:

In the *first* quadrant, as θ increases from 0° to 90°,
$\sin\theta$ *increases from* 0 *to* 1.

In the *second* quadrant, as θ increases from 90° to 180°,
$\sin\theta$ *diminishes from* 1 *to* 0.

In the *third* quadrant, as θ increases from 180° to 270°,
$\sin\theta$ *diminishes from* 0 *to* -1.

In the *fourth* quadrant, as θ increases from 270° to 360°,
$\sin\theta$ *increases from* -1 *to* 0.

(ii) Changes in cosine.

In the first quadrant, as the angle XOP_1 increases, ON_1 diminishes, from the value of OX at $\theta = 0^\circ$ to the value 0 at $\theta = 90^\circ$, and is always positive.

In the second quadrant, as θ goes on increasing from 90° to 180°, ON_3 increases numerically from 0 to OX' but is

negative. In the third quadrant, ON_4 remains negative, but diminishes numerically from OX' to 0. In the fourth quadrant, ON_5 is positive and increases from 0 to OX again.

The hypotenuse remains always positive and is equal to OX or OX' in magnitude.

We thus come to the conclusions :

As θ increases *from* 0° *to* 90°,

cos θ *diminishes from* 1 *to* 0.

As θ increases *from* 90° *to* 180°,

cos θ *diminishes from* 0 *to* − 1.

As θ increases *from* 180° *to* 270°,

cos θ *increases from* − 1 *to* 0.

As θ increases *from* 270° *to* 360°,

cos θ *increases from* 0 *to* 1.

(iii) Changes in tangent.

As θ goes on increasing from 0° to 90° in the first quadrant, P_1N_1 increases from 0 to OY and simultaneously ON_1 decreases from OX to 0, both remaining positive ; hence, $\tan\theta = \dfrac{P_1N_1}{ON_1}$ increases from the value $\dfrac{0}{OP} = 0$ to $\dfrac{ON}{0} \rightarrow \infty$.

In the second quadrant, P_3N_3 diminishes from OY to 0 while ON_3, becoming negative, numerically increases from 0 to OX'. Hence, $\tan\theta = \dfrac{P_3N_3}{ON_3}$ is negative but numerically diminishes from ∞ to 0, *i.e.*, increases from $-\infty$ to 0.

Immediately before 90°, tan θ is positive and very large, while immediately after 90°, tan θ is negative and numerically very large. In fact, here, as θ passes through the value 90° from the first to the second quadrant, there is

a sudden break or discontinuity in the value of tan θ, which suddenly passes from a very large positive value to a very large negative value, *i.e.*, from the positive to the negative side in passing through infinity.

In the third quadrant, both P_4N_4 and ON_4 are negative and P_4N_4 increases numerically from 0 to OY' while ON_4 decreases numerically from OX' to 0. Hence, $\tan\theta = \dfrac{P_4N_4}{ON_4}$ is positive, and increases from 0 to ∞.

In the fourth quadrant, P_5N_5 is negative and numerically diminishes from OY' to 0 while ON_5 is positive and increases from 0 to OX. Hence, $\tan\theta = \dfrac{P_5N_5}{ON_5}$ is negative and numerically diminishes from ∞ to 0, *i.e.*, increases from $-\infty$ to 0.

In passing through 270°, there is another discontinuity, tan θ suddenly passing from the positive to the negative side through infinity.

The results are therefore as follows :

As θ increases *from* 0° *to* 90°, tan θ *increases from* 0 *to* ∞

As θ passes through 90°, tan θ suddenly changes from $+\infty$ to $-\infty$

As θ increases *from* 90° *to* 180°, tan θ *increases from* $-\infty$ *to* 0

As θ increases *from* 180° *to* 270°, tan θ *increases from* 0 *to* ∞

As θ passes through 270°, tan θ suddenly changes from $+\infty$ to $-\infty$

As θ increases *from* 270° *to* 360°, tan θ *increases from* $-\infty$ *to* 0.

(iv) Changes in cotangent.

From the changes in the value of the tangent the changes in $\cot\theta = \frac{1}{\tan\theta}$ are traced as follows :

θ increasing *from* $0°$ *to* $90°$, $\cot\theta$ *diminishes from* ∞ *to* 0

θ increasing *from* $90°$ *to* 180, $\cot\theta$ *diminishes from* 0 *to* $-\infty$

As θ passes through $180°$, there is a sudden change in $\cot\theta$ from $-\infty$ to $+\infty$

θ increasing *from* $180°$ *to* $270°$, $\cot\theta$ *diminishes from* $+\infty$ *to* 0

θ increasing *from* $270°$ *to* $360°$, $\cot\theta$ *diminishes from* 0 *to* $-\infty$

As θ passes through $360°$, $\cot\theta$ again suddenly changes from $-\infty$ to $+\infty$.

(v) Changes in secant.

For $\sec\theta = \frac{1}{\cos\theta}$, the results are as follows :

From $0°$ *to* $90°$ for θ, $\sec\theta$ *increases from* 1 *to* ∞.
Here, there is a sudden change from $+\infty$ to $-\infty$.
Then *from* $90°$ *to* $180°$, $\sec\theta$ *increases from* $-\infty$ *to* -1.
From $180°$ *to* $270°$, $\sec\theta$ *diminishes from* -1 *to* $-\infty$.
Here, again there is a sudden change from $-\infty$ to $+\infty$.
Then *from* $270°$ *to* $360°$, $\sec\theta$ *diminishes from* ∞ *to* 1.

(vi) Changes in cosecant.

For $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$, the results are as follows :

From $0°$ *to* $90°$, $\operatorname{cosec}\theta$ *diminishes from* ∞ *to* 1.
From $90°$ *to* $180°$, $\operatorname{cosec}\theta$ *increases from* 1 *to* ∞.
Here, $\operatorname{cosec}\theta$ suddenly changes from $+\infty$ *to* $-\infty$.

Then *from* 180° *to* 270°, cosec θ *increases from* $-\infty$ *to* -1.

From 270° *to* 360°, cosec θ *diminishes from* -1 *to* $-\infty$.

As θ passes through 360°, cosec θ again suddenly changes from $-\infty$ to $+\infty$.

Note. As θ increases by complete multiples of 2π (*i.e.*, 360°) we know that all the Trigonometrical ratios remain unaltered. Hence after 360°, as θ goes on increasing, the same series of values for the ratios are repeated over and over again for each complete revolution of the revolving line. The trigonometrical ratios are therefore all of them **periodic functions** having the same **period** 2π,* after each of which the same cycle of values is repeated.

The changes traced out above, of the trigonometrical ratios, may be much more clearly demonstrated to the eye from a study of their graphs.

103. Graphs of Trigonometrical Functions.

Just like algebraic functions, trigonometrical functions (*i.e.*, $\sin x$, $\cos x$, $\sin^2 2x + \tan\frac{x}{2}$, etc.) may be conveniently represented by means of graphs, showing their changes with the change in the values of the angles.

The method is the same as for graphs in Algebra. Two straight lines XOX' and YOY', intersecting at right angles are taken as axes of co-ordinates. Along the x-axis, the angles are represented on a suitably chosen scale, positive angles along OX, and negative angles along OX'. Along the y-axis the values of the trigonometrical functions corresponding to the angles are represented on a suitably chosen scale, positive values being measured upwards (along OY), and negative values downwards (along OY'). Thus, the *abscissa* and *ordinate* of a point stand respectively for an angle and its trigonometrical function.

* tan θ and cot θ have a period π.

Plotting a number of points in this way and joining them free-hand, we get the required graph of a given trigonometrical function.

104. Graph of six x *or sine-graph.*

Let $y = \sin x$.

Using the table of natural sines, the corresponding values of x and y are tabulated corresponding to the values of x differing by 10° (the values of y being correct up to two places of decimals) as follows :—*

x	-90°	-80°	-70°	-60°	-50°	-40°	-30°	-20°	-10°	0°
y or $\sin x$	-1	$-\cdot98$	$-\cdot94$	$-\cdot87$	$-\cdot77$	$-\cdot61$	$-\cdot50$	$-\cdot34$	$-\cdot17$	0

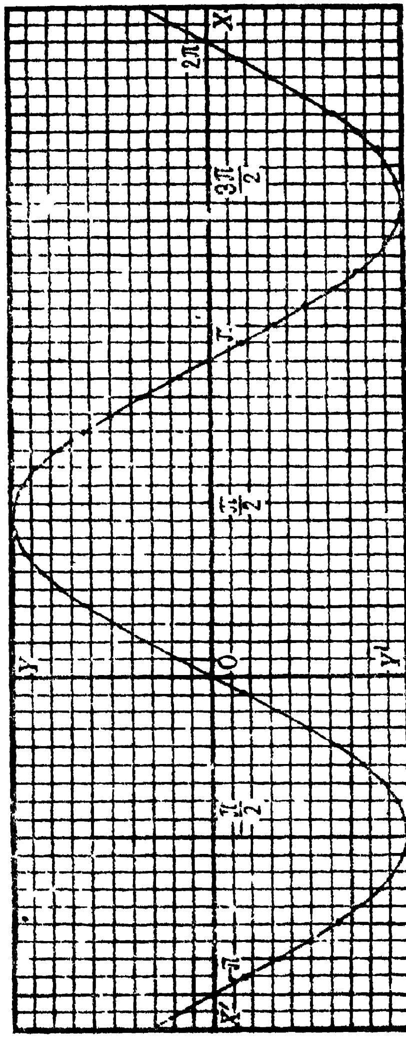
x	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	110°	120°	etc.
y or $\sin x$	$\cdot17$	$\cdot34$	$\cdot50$	$\cdot64$	$\cdot77$	$\cdot87$	$\cdot94$	$\cdot98$	1	$\cdot98$	$\cdot94$	$\cdot87$	etc.

Now, let the scale be so chosen that 1 small division along OX represents 10°, and 10 small divisions along OY represent unity.*

The points corresponding to the tabulated values are plotted on the graph paper according to the scale chosen and joined free-hand.

The graph is as shown on the next page (drawn here between the range $x = -180^\circ$ to $x = +360^\circ$).

*According to the graph paper supplied and the range within which the graph is to be drawn, the scale should be suitably chosen in each individual case separately.



Sine-graph

Note 1 In the table of natural sines, sines of angles from 0° to 90° only are available With the help of the formulæ $\sin(-\theta) = -\sin\theta$, $\sin(180° - \theta) = \sin\theta$ $\sin(180° + \theta) = -\sin\theta$ etc of Chapter IV however, the tabulation for $\sin\theta$ shown above outside the range of 0° to 90°, is effected

Similar is the case of tabulation for other graphs in the following pages

Note 2 *The characteristic features of the graph.*

From the figure the following features will be apparent —(i) the graph is continuous and wavy in form (ii) the maximum value of $\sin x$ is +1 and the minimum value is −1 these values being attained for values of x which are odd multiples of 90° (iii) $\sin x$ is 0 at the origin and at points for which x is an even multiple of 90° *i.e.*, any multiple of 180° (iv) that $\sin\left(\frac{\pi}{2} - x\right) = \sin\left(\frac{\pi}{2} + x\right)$, $\sin(\pi - x) = \sin x$ $\sin(-x) = -\sin x$ $\sin(\pi + x) = -\sin x$ etc (v) since $\sin(2n\pi + x) = \sin x$ the portion between 0 to 2π is repeated over and over again on either side

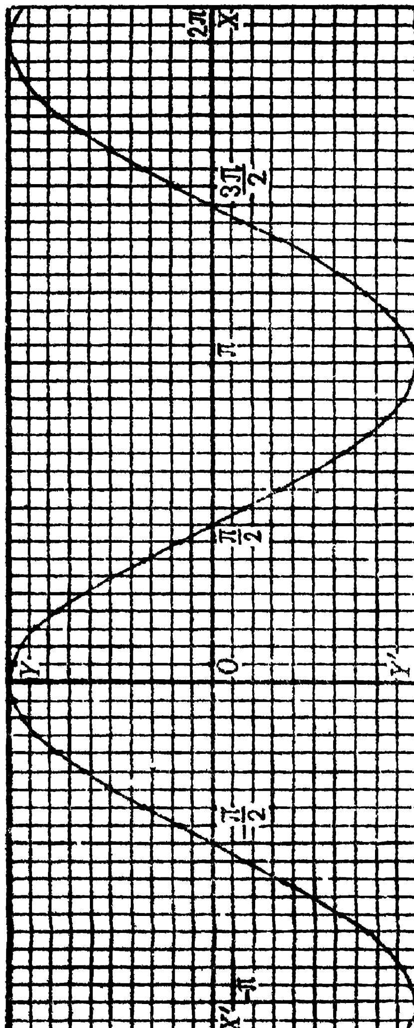
105. Graph of cos x *or cosine-graph*

Let $y = \cos x$

Using the table of natural cosines (see *Note 1 of the previous Article*), the corresponding values of x and y are tabulated at intervals of 10° for x as follows —

x	−90°	−80°	−70°	−60°	−50°	−40°	−30°	−20°	−10°
y or $\cos x$	0	·17	·34	·50	·64	·77	·87	·94	·98

x	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	110°	etc
y or $\cos x$	1	·98	·94	·87	·77	·64	·50	·34	·17	0	−·17	−·34	etc.



Cosine-graph

Now, choosing the scale such that 1 small division along OX represents 10°, and 10 small divisions along OY represent unity, the points corresponding to the above tabulated values are plotted and joined free-hand.

We then get the required graph, which is shown on the annexed page (shown here between the range $-\pi$ to $+2\pi$ of x).

Note. It is apparent from the figure, that the *cosine graph is exactly the same as the sine-graph only shifted wholesale backwards (to the left) through a space of* 90°.

This is due to the fact that $\sin(90^\circ+x)=\cos x$, or $\sin x=\cos(x-90^\circ)$ so that the ordinate in the sine-graph corresponding to any value of x = the ordinate of the cosine-graph corresponding to a value of x which is 90° less than before.

106. Graph of tan x or *tangent-graph*.

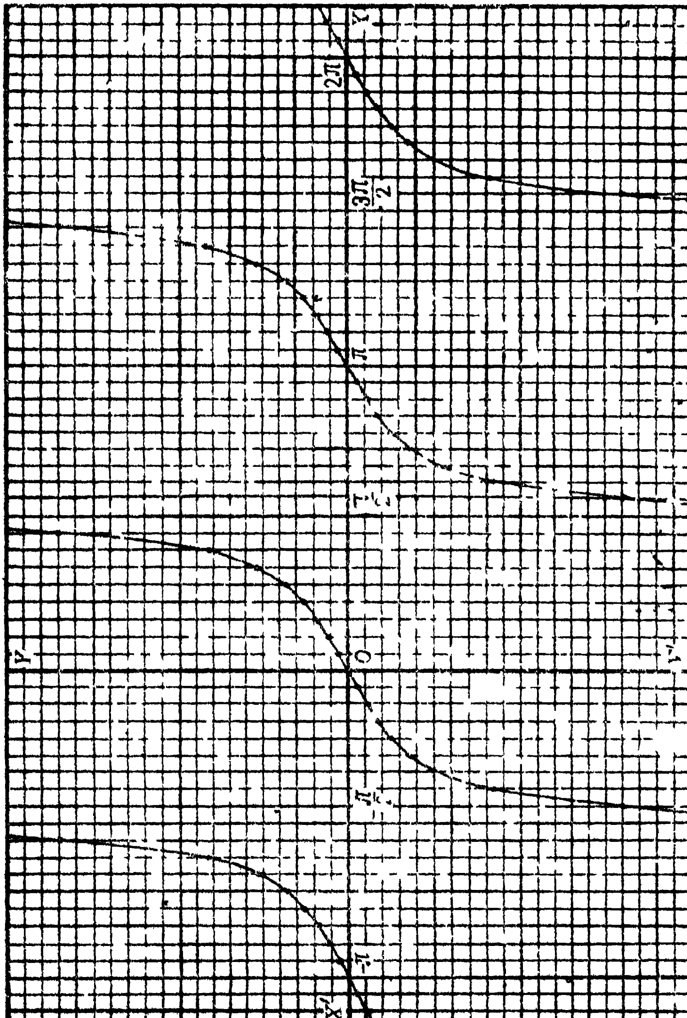
Let $y=\tan x$.

Using the table of natural tangents, the corresponding values of x and y are tabulated at intervals of 10° of x as follows :—

x	-20°	-10°	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	etc.
y or $\tan x$	$-\cdot 36$	$-\cdot 18$	0	$\cdot 18$	$\cdot 36$	$\cdot 58$	$\cdot 84$	$1\cdot 19$	$1\cdot 73$	$2\cdot 75$	$5\cdot 67$	∞	$-5\cdot 67$	etc.

Now, choosing the scale such that 1 small division along OX represents 10°, and 3 small divisions along OY represent unity, the points corresponding to the above tabulated values are plotted and joined free-hand.

The graph is as shown on the next page (shown here between the range $-\pi$ to $+2\pi$ for x).



Tangent-Graph

Note. *Peculiarities of the tangent graph.*

From the figure, the following points will be apparent: (i) That the curve is not continuous, but consists of separate branches or portions, the points of discontinuities being the values of x corresponding to the odd multiples of $\frac{\pi}{2}$. (ii) As x passes through these points from the left to the right, the value of $\tan x$ suddenly changes from very large positive values on the left to very large negative values on the right. (iii) The lines parallel to y-axis corresponding to the odd multiples of $\frac{\pi}{2}$ are continuously approached by the graph on either side, but never actually met. Such lines are called *asymptotes* to the curve. (iv) Since $\tan(\pi + x) = \tan x$, each branch is simply a repetition of the branch from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$.

107. Graph of cot x or *cotangent-graph.*

As before the values of x and y ($= \cot x$) are tabulated, and with the same scale as in the tangent-graph the points are plotted and joined free-hand.

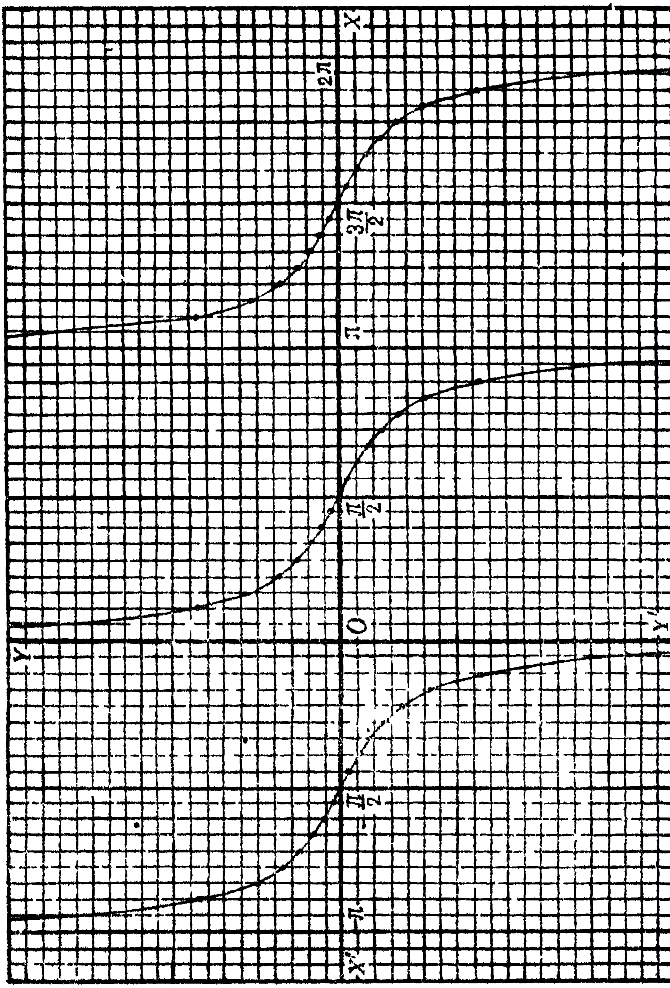
The graph is shown on the next page (between $x = -\pi$ to $x = +2\pi$).

This graph also, like the tangent-graph, is *discontinuous*, the points of discontinuity being $x = 0$ and $x = n\pi$. The portion between $x = 0$ and $x = \pi$ is repeated over and over again on either side, as is consequent from the formula $\cot(n\pi + x) = \cot x$.

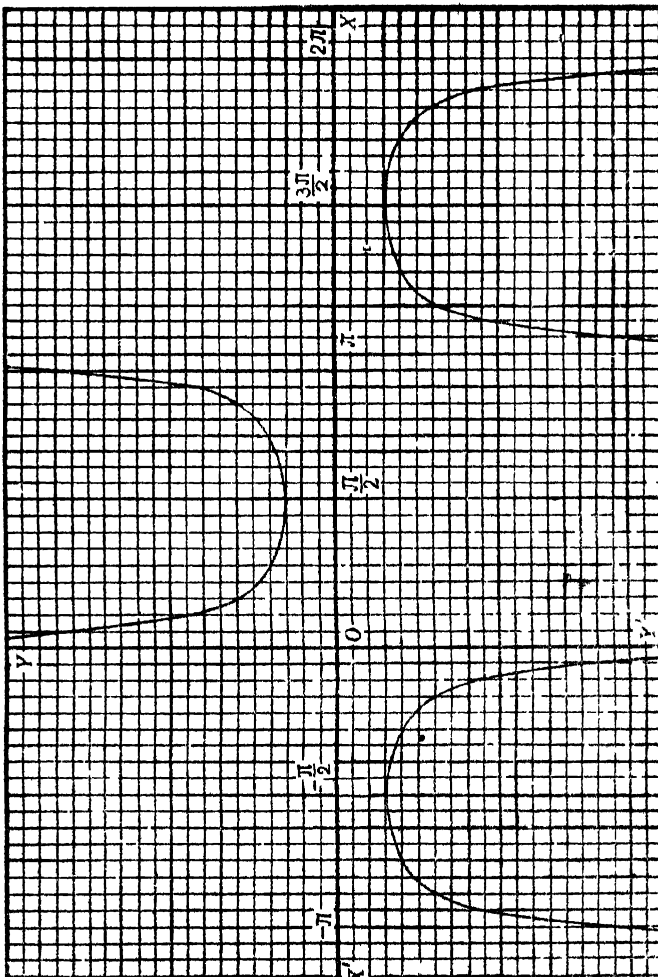
108. Graph of cosec x or *cosecant-graph.*

The corresponding values of x and y are tabulated at intervals of 10° of x as follows :—

x	-20°	-10°	0°	10°	20°	30°	etc.	80°	90°	100°	110	etc.
y or cosec x	$-2{\cdot}92$	$-5{\cdot}76$	∞	$5{\cdot}76$	$2{\cdot}92$	2	etc.	$1{\cdot}02$	1	$1{\cdot}02$	$1{\cdot}06$	etc.



Cotangent-Graph



Cosecant-Graph

[If the table of natural cosecants be not available, the table of natural sines may be used and the values of $\operatorname{cosec} x = \dfrac{1}{\sin x}$ may be calculated for different values of x.]

The scale is so chosen that 1 small division along OX represents 10°, and 3 small divisions along OY represent unity.

The tabulated points are now plotted and joined free-hand.

The graph is shown on the previous page (between the range $x = -\pi$ to $x = 2\pi$.)

Note 1. This graph also consists of *detached branches*, the points of discontinuity being $x = 0$ and $x = n\pi$. The value of y never lies between ± 1, being always greater than 1 or less than -1. The lines $x = n\pi$ are asymptotes. The portion between $x = 0$ to $x = 2\pi$ is repeated on either side, over and over again.

109. Graph of sec x or *secant-graph.*

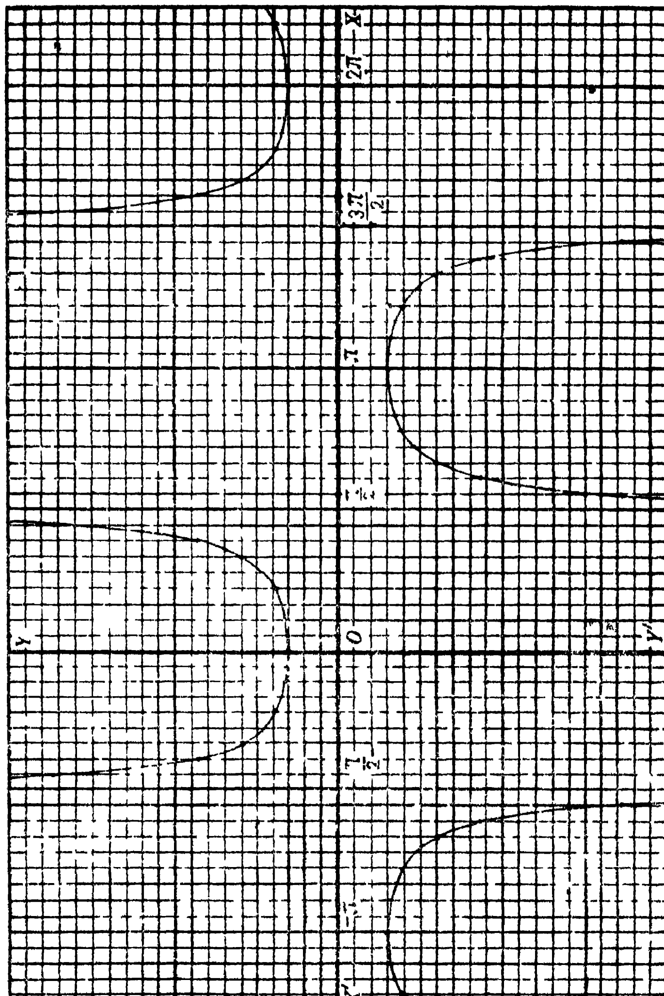
The corresponding values of x and y ($= \sec x$) are tabulated as in the case of co-secant-graph, by making use of the table of cosines, if a table of secants be not available.

With the same scale as in the cosecant-graph, the tabulated points have been plotted and joined free-hand.

The graph is shown in the adjoining page (between the range $x = -\pi$ to $x = 2\pi$.)

Note. It is apparent from the figure that the *secant-graph is exactly the same as the cosecant-graph, only shifted backwards (to the left) through a space of* 90°.

This is due to the fact that $\operatorname{cosec}(90° + x) = \sec x$. [See *note below Art. 105*]



Secant-Graph

110. Graphs of other Trigonometrical Expressions.

Graphs of other trigonometrical functions may be obtained in a similar manner.

We illustrate this by an example.

Ex. *Draw the graph of $y = \sin x + \cos x$ between the range $x = 0$ to $x = 2\pi$, and find from the graph the values of x for which (i) $y = 0$, (ii) y is maximum, (iii) y is minimum.*

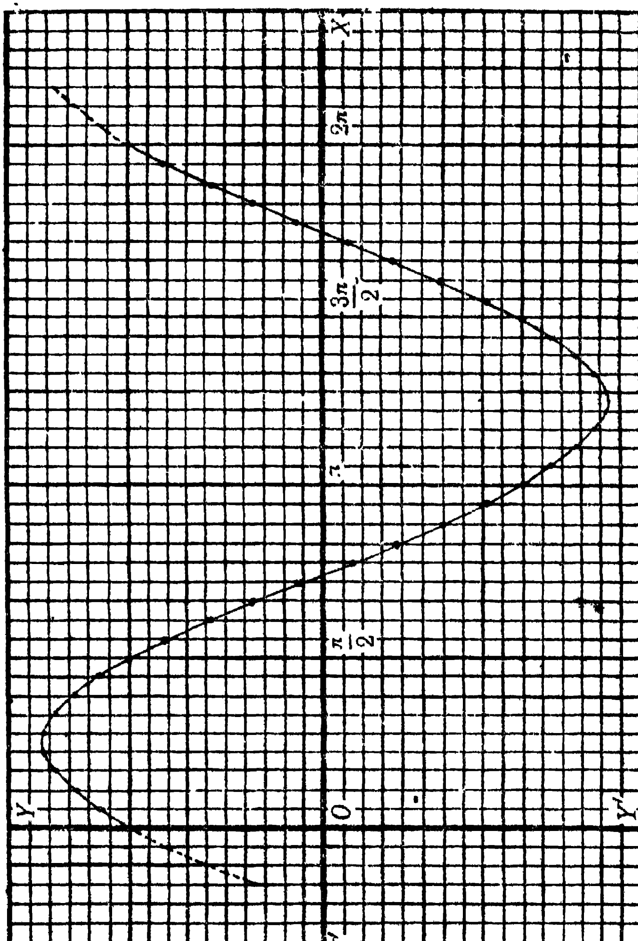
[*C. U. 1931*]

From the table of natural sines and cosines, corresponding to each value of x, the values of $\sin x$ and $\cos x$ may be separately obtained and then added to give y, or else we may write $y = \sin x + \cos x = \sqrt{2}(\sin x \cos \frac{1}{4}\pi + \cos x \sin \frac{1}{4}\pi)$ $= \sqrt{2} \sin (x + \frac{1}{4}\pi)$, and corresponding to any value of x, $\sin (x + \frac{1}{4}\pi)$ may be deduced from the sine-table, and this multiplied by $\sqrt{2}$ ($= 1\cdot414$) will give y.

The corresponding values of x and y are tabulated at intervals of 10° of x, between the interval $x = 0$ to $x = 2\pi$ as follows :—

x	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°
y	1	1·15	1·27	1·37	1·41	1·41	1·37	1·27	1·15	1	·81

x	110°	120°	130°	140°	150°	160°	170°	180°	190°	200°
y	·59	·37	·13	−·13	−·37	−·59	−·81	−1	−1·15	−1·27



Graph of $\sin x + \cos x$

x	210°	220°	230°	240°	250°	260°	270°	280°
y	−1·37	−1·41	−1·41	−1·37	−1·27	−1·15	−1	−·81

x	290°	300°	310°	320°	330°	340°	350°	360°
y	−·59	−·37	−·13	·13	·37	·59	·81	1

The scale is chosen so that 1 small division along OX represents 10°, and 10 small divisions along OY represent unity.

The tabulated points are now plotted and joined. The graph is as shown on the previous page

From the graph we find that (i) $y = 0$ when $x = 135°$ and $x = 315°$, (ii) y is maximum when $x = 45°$, (iii) y is minimum when $x = 225°$

111. Graphical solution of Equations.

Trigonometrical equations, just like algebraic equations may be solved graphically In fact in many practical cases, particularly where the solutions are not obtained in terms of the standard angles the graphical method of solution is the only one which is found convenient and is actually adopted. The method is illustrated by the following examples.

Ex. 1. *Solve graphically the equation* $2 \sin^2 x = \cos 2x$, *giving only those solutions of x which lie between* $-\dfrac{\pi}{2}$ *and* $\dfrac{3\pi}{2}$.

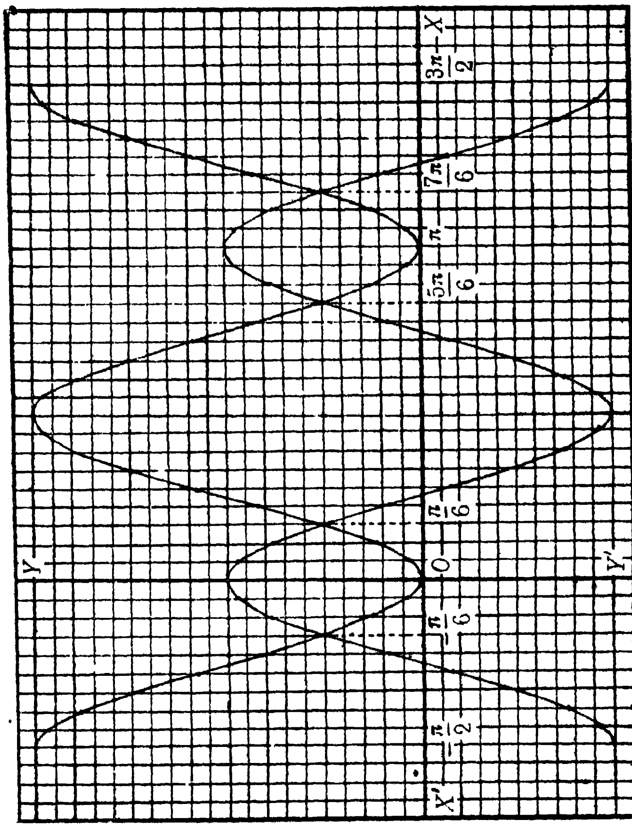
[*C. U. 1938, '46, '48*]

We draw two graphs, namely

$$y = 2 \sin^2 x \; (= 1 - \cos 2x)$$

$$\text{and} \quad y = \cos 2x$$

by tabulating the corresponding values of x and y for the two cases separately, making use of the table of natural.



Graphical solution of $2\sin^2 x = \cos 2x$.

cosines, for the range $x = -\frac{\pi}{2}$ to $x = \frac{3\pi}{2}$, at intervals of 10° or 15° of x.

With the same scale, namely, 1 small division along OX representing 10°, and 10 small divisions along OY representing unity, we plot the tabulated values for the two cases in the same graph paper, and joining them, we get the two graphs, as shown in the adjoining page.

We find that the two graphs intersect, and thus have the same abscissæ and ordinates at the points for which

$$x = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}.$$

Thus, $2 \sin^2 x = \cos 2x$ is satisfied for the values of x given by

$$x = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6} \text{ and } \frac{7\pi}{6}$$

which are the required solutions within the range

$$-\frac{\pi}{2} \text{ to } \frac{3\pi}{2}.$$

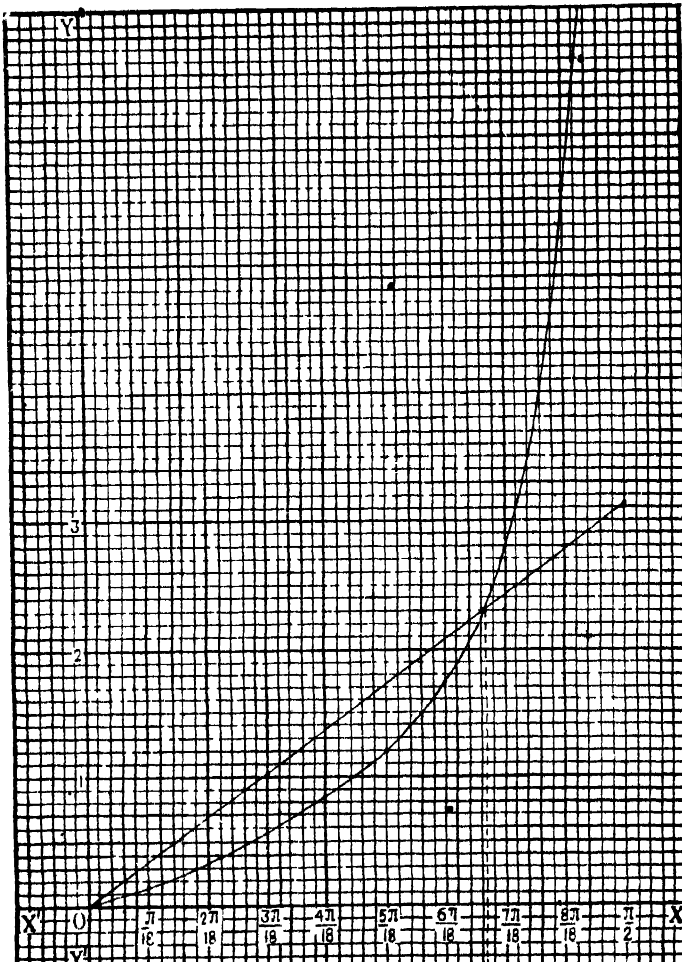
Ex. 2. *Solve graphically the equation* $\tan x = 2x$ *between* $x = 0$ *and* $x = \frac{\pi}{2}$. [*C. U. 1939*]

Here, x is supposed to be measured in radians.

First of all we draw separately two graphs, namely

$$y = 2x \quad \cdots \quad \cdots \quad \text{(i)}$$

and $$y = \tan x \quad \cdots \quad \cdots \quad \text{(ii)}$$



Graphical solution of $\tan x = 2x$.

The corresponding values of x and y within the range $x=0$ and $x=\frac{\pi}{2}$ are tabulated in case (i) as follows :

x (in radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y (i.e. $2x$) (numerical value)	0	1·05	2·10	3·15

and in case (ii) as follows :

x (in radians)	0	$\frac{\pi}{18}$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$	$\frac{7\pi}{18}$	$\frac{8\pi}{18}$	$\frac{\pi}{2}$
y (i.e. $\tan x$) (numerical value)	0	·18	·36	·57	·84	1·19	1·73	2·75	5·67	∞

Now, choosing the same scale, namely 5 small divisions along OX to represent $\frac{\pi}{18}$ radians, and 10 small divisions along OY to represent unity, we plot the tabulated points for the two cases in the same graph paper and joining them we get the two graphs within the range $x=0$ and $x=\frac{\pi}{2}$ as shown in the adjoining page.

We find that the two graphs intersect at the point given by $x=0$ and also at the point corresponding to 33·5 small divisions along OX, which, from our chosen scale, represents $x=\frac{33\cdot5}{5}\times\frac{\pi}{18}$ radians $=$ 1·17 radians (approximately).

Hence, the given equation $\tan x=2x$ is satisfied between $x=0$ and $x=\frac{\pi}{2}$ by the values of x, namely $x=0$ and $x=1\cdot17$ (approximately), which are the required solutions in radians.

Examples XVI

1. Draw the graphs of

(i) $\sin 3x$ between $x = 0°$ to $x = 180°$.

(ii) $\tan \frac{3}{2}x$ between $x = -\frac{1}{2}\pi$ to $x = \pi$.

(iii) $\sin \theta \cos \theta$ between $\theta = -\pi$ to $\theta = +\pi$.

(iv) $\dfrac{1}{\cos^2\theta - \sin^2\theta}$ between $\theta = -\dfrac{\pi}{2}$ to $+\dfrac{\pi}{2}$.

(v) $\cos (\pi \sin x)$ between $x = 0$ to $x = \frac{1}{2}\pi$.

(vi) $\sin \theta - \sqrt{3} \cos \theta$ between $\theta = 0$ to $\theta = \pi$.

(vii) $\frac{1}{2} \operatorname{cosec} \frac{1}{2}x$ between $x = 0$ to $x = 2\pi$.

2. (i) Trace the changes in the sign of $\cos \theta - \sin \theta$ as θ changes from $0°$ to $360°$. Verify your conclusions by a graph.

(ii) Trace the changes in sign and magnitude of
$$\frac{2 \sin \theta - \sin 2\theta}{2 \sin \theta + \sin 2\theta}.$$
[*B. H. U. 1931*]

3. Draw the graph of $y = \sin (x + \frac{1}{2}\pi)$ between the limits $x = -\pi$ and $x = +\pi$.

4. Draw the graphs of $\sin \theta$ and $\cos \theta$ between $\theta = 0$ and $\theta = \pi$. Find the points where the graphs intersect.
[*C. U. 1936, '46*]

5. Construct the graphs of $\tan x$ and $\cos x$ between 0 and $\frac{1}{2}\pi$ for x, making a tabulation of the values of y dividing the interval into 9 equal parts.

If $\tan x = \cos x$, find approximately the value of x from the above two graphs. [*C. U. 1945*]

6. Obtain graphically a solution of the equation $\tan x = 1$, between $x = 0$ and $x = \frac{1}{2}\pi$. [*C. U. 1937*]

[*Draw the graphs of $y = \tan x$ and $y = 1$.*]

7. Draw the graph of $\cos x - \sin 2x$ for values of x lying between 0° and 90° and hence obtain the least value of $\cos x - \sin 2x$ in this range.

8. Solve graphically the equations

(i) $x - \tan x = 0$, between $x = 0$ and $x = \frac{1}{2}\pi$

[C. U. 1915]

(ii) $5 \sin \theta + 2 \cos \theta = 5$, between $\theta = 0^\circ$ and $\theta = 270^\circ$

[*Draw the graphs of $y = 5 \sin \theta + 2 \cos \theta$ and $y = 5$ and find the common points.*] [C. U. 1947]

(iii) $\cot \theta - \tan \theta = 2$ between $\theta = 0$ and $\theta = \pi$

[C. U. 1919]

(iv) $\operatorname{cosec} x = \cot x + \sqrt{3}$, between $x = 0$ and $x = \pi$

(v) $\cos x = \sin 2x + \frac{1}{2}$ between $x = -\frac{1}{2}\pi$ and $x = +\frac{1}{2}\pi$

(vi) $5 - \tan x = 2x$, between 0 and 2π

(vii) $2 \sin x + x - 3 = 0$

(viii) $x^2 = \cos x$

(ix) $x = \cos^2 x$

[*Draw the graph of $y = \cos 2x$ and $y = 2x - 1$.*]

9. Represent by a graph the displacement given by
$s = 2 \sin t + \sin 3t$

CHAPTER XVII

HARDER PROBLEMS ON HEIGHTS AND DISTANCES

112. Some simple practical applications of Trigonometry, dealing with easy problems on determination of heights and distances, have already been discussed in Chapter V. The problems in the present section are of a more general character, requiring for their solutions, the general relations between the sides and angles of a triangle, as also some geometrical skill.

113. *To find the height and the distance of an inaccessible object standing on a horizontal plane.*

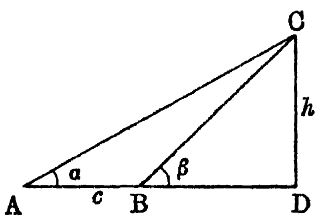


Fig. (i)

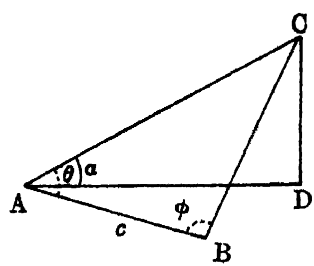


Fig. (ii)

Let CD be the object, which is observed from a point A on a horizontal ground, α being the observed elevation of its top C. Let h be the required height CD and d the required distance AD of the object from A.

Case I. If possible, measure off any suitable distance AB $(=c)$ from A directly towards the object, and from B let the observed elevation of C be β.

Then, from fig. (i),

$$c = AD - BD = h \cot \alpha - h \cot \beta$$

$$= h\left(\frac{\cos \alpha}{\sin \alpha} - \frac{\cos \beta}{\sin \beta}\right) = \frac{h \sin (\beta - \alpha)}{\sin \alpha \sin \beta}.$$

$\therefore \quad h = c \sin \alpha \sin \beta \operatorname{cosec} (\beta - \alpha).$

Also $d = AD = h \cot \alpha = c \cos \alpha \sin \beta \operatorname{cosec} (\beta - \alpha).$

Note. Each of the above expressions for h and d is in a suitable form for logarithmic computation.

Case II. If however it is not convenient to measure the length AB directly towards the object, we may proceed as follows :

Measure off the length $AB(=c)$ in *any* suitable direction from A. From A let the observed elevation of C be α as before. The angles CAB and CBA are also observed from A and B respectively. Let these be θ and ϕ.

We get from fig. (ii) in this case,

$$\text{in } \triangle ABC, \ \frac{AC}{\sin \phi} = \frac{AB}{\sin C},$$

$$i.e., \quad = \frac{c}{\sin (180^\circ - \theta - \phi)} = \frac{c}{\sin (\theta + \phi)}.$$

$\therefore \quad AC = c \sin \phi \operatorname{cosec} (\theta + \phi).$

$\therefore \quad h = AC \sin \alpha = c \sin \alpha \sin \phi \operatorname{cosec} (\theta + \phi)$

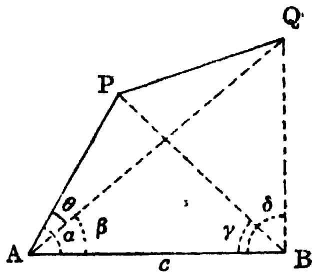
and $\quad d = AD = AC \cos \alpha = c \cos \alpha \sin \phi \operatorname{cosec} (\theta + \phi).$

Note. Here also, the expressions for h and d are suitable for calculation by logarithm.

114. *To find the distance between two visible but inaccessible objects.*

Let P and Q be the objects whose distance apart is required.

Take two suitable points A and B for observation, the distance between which is measured, say c.



At A, measure the angles PAQ, PAB, and QAB (the second observation being unnecessary if all the four points P, A, B, Q are in the same plane, for in that case, $\angle PAB = \angle PAQ + \angle QAB$). Let these be θ, α and β respectively.

At B, measure the angles PBA and QBA, and let them be γ and δ.

From $\triangle PAB$, $\dfrac{PA}{\sin\gamma} = \dfrac{c}{\sin(180^\circ - \alpha - \gamma)} = \dfrac{c}{\sin(\alpha+\gamma)}$,
whence, $PA = c \sin\gamma \operatorname{cosec}(\alpha+\gamma)$.

Similarly, from $\triangle QAB$,

$$QA = c \sin\delta \operatorname{cosec}(\beta+\delta).$$

Lastly, from $\triangle PAQ$,

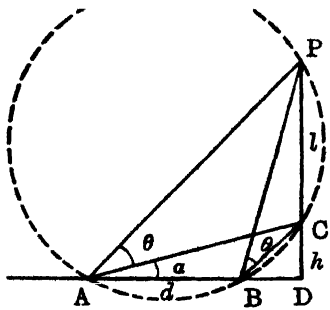
$$PQ^2 = PA^2 + QA^2 - 2PA \,.\, QA \,.\, \cos\theta.$$

Thus, PQ is determined.

115. Some more illustrative examples of harder problems on heights and distances are worked out below.

Ex. 1. *A flagstaff is fixed on the top of a tower standing on a horizontal plane. An observer walking directly towards the foot of the tower, observes the angle subtended*

by the flagstaff from two positions on his path to be the same namely θ. The distance between these two positions is d, and the angle subtended by the tower at his first position is α. Find the height of the tower, and the length of the flagstaff.



Let CD be the tower, PC the flagstaff, whose heights required are h and l respectively. A and B are the points of observation.

$\because$ $\angle PAC = \angle PBC = \theta$, the points P, A, B, C are concyclic,

$$\therefore \quad \angle CBD = \angle APC = 90^\circ - \angle PAD = 90^\circ - (\theta + \alpha).$$

Now,
$$d = AD - BD = h \cot \alpha - h \cot (CBD)$$
$$= h \{\cot \alpha - \tan (\theta + \alpha)\}$$
$$= h \left\{\frac{\cos \alpha}{\sin \alpha} - \frac{\sin (\theta + \alpha)}{\cos (\theta + \alpha)}\right\} = h \frac{\cos (\theta + 2\alpha)}{\sin \alpha \cos (\theta + \alpha)}.$$

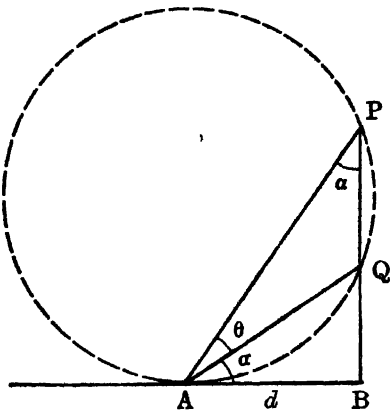
$$\therefore \quad h = d \sin \alpha \cos (\theta + \alpha) \sec (\theta + 2\alpha).$$

Again, from $\triangle APC$,

$$\frac{l}{\sin \theta} = \frac{AC}{\sin APC} = \frac{h}{\sin \alpha \cos (\theta + \alpha)} = \frac{d}{\cos (\theta + 2\alpha)}.$$

$$\therefore \quad l = d \sin \theta \sec (\theta + 2\alpha).$$

• **Ex. 2.** *A man walking towards a building, on which a flagstaff is fixed, observes the angle subtended by the flagstaff to be greatest, when he is at a distance d from the building. If θ be the observed greatest angle, find the length of the flagstaff, and the height of the building.*



Let QB be the building, and PQ the flagstaff. It is easily seen from Geometry that the point of contact A of a circle through P and Q touching the path of the observer on the ground, is the point at which the angle subtended by PQ is the greatest.

Thus, $\angle QAB = \angle APQ = \alpha$ say.

Then, $\angle PAB + \angle APB = 90^\circ$,

or $\theta + 2\alpha = 90^\circ$. $\qquad\cdots$ (i)

Now, $PQ = PB - QB = d\tan(\theta+\alpha) - d\tan\alpha$

$$= d\left\{\frac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)} - \frac{\sin\alpha}{\cos\alpha}\right\}$$

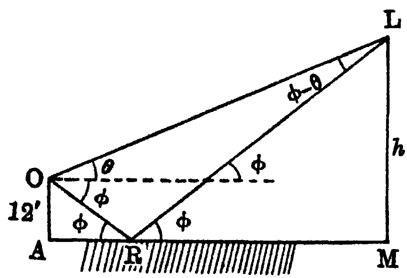
$$= d\frac{\sin\theta}{\cos(\theta+\alpha)\cos\alpha} = \frac{2d\sin\theta}{\cos(\theta+2\alpha)+\cos\theta}$$

$$= 2d\tan\theta. \qquad [\textit{from (i)}]$$

Also, $QB = d\tan\alpha = d\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$.

Ex. 3. *The angle of elevation of a light at the top of a distant tower from a point* 12 *ft. above a lake is* 24° 55′, *and the angle of depression of its reflection in the lake is* 35° 5′. *Find the height of the tower correct to two decimal places, having given*

$$\log 2 = \cdot 30103, \qquad \log 3 = \cdot 47712$$
$$\log 588 = 2\cdot 76938, \qquad \log 589 = 2\cdot 77012$$
$$L \sin 10^\circ\, 10' = 9\cdot 24677.$$



Let L be the light at the top of the tower LM, LRO the ray from L, which reflected in the lake at R, reaches the observer O, so that OR is the direction in which the reflexion is seen, and thus from the laws of reflexion, $\angle ORA = \angle LRM = \phi$ (say) which is evidently equal to the angle of depression of the reflexion, *i.e.*, 35° 5′.

Let θ denote the angle of elevation of L from O, *i.e.*, 24° 55′.

Now, from the figure, in $\triangle ORL$,

$$\frac{RL}{\sin(\theta+\phi)} = \frac{OR}{\sin(\phi-\theta)} = \frac{12}{\sin\phi\,\sin(\phi-\theta)} \text{ ft.}$$

$$\therefore \quad h = LM = RL \sin\phi = 12\,\frac{\sin(\theta+\phi)}{\sin(\phi-\theta)} = 12\,\frac{\sin 60^\circ}{\sin(10^\circ\, 10')}$$

$$= \frac{6\sqrt{3}}{\sin(10^\circ\, 10')} = \frac{2.3^{\frac{3}{2}}}{\sin(10^\circ\, 10')}.$$

Hence, $\log h = \log (2.3^{\frac{3}{2}}) - \log \sin (10°10')$

$$= \log 2 + \tfrac{3}{2} \log 3 + 10 - L \sin (10°10')$$

$$= \cdot 30103 + \tfrac{3}{2}(\cdot 47712) + 10 - 9\cdot 24677$$

$$= 1\cdot 76994.$$

From the given data, it is seen that

$\log h$ lies between $\log 58\cdot 8$ and $\log 58\cdot 9$.

Hence, if $h = 58\cdot 8 + x$, diff. for $\cdot 1 = 1\cdot 77012 - 1\cdot 76938$
$= \cdot 00074$,

and diff. for $x = 1\cdot 76994 - 1\cdot 76938 = \cdot 00056$.

$\therefore$ by the theory of proportional parts,

$$\frac{x}{\cdot 1} = \frac{56}{74} = \cdot 75 \quad \therefore \quad x = \cdot 075 = \cdot 08 \text{ approximately.}$$

Thus, $h = 58\cdot 88$ ft.

Examples XVII

1. The angle of elevation of the top of a palm tree standing on horizontal ground, observed from two points A and B, distant 40 and 30 feet from the foot, and in the same straight line with it are found to be complementary. Prove that the height of the tree is nearly 35 feet, and that the angle subtended at the top of the tree by the line AB is $\sin^{-1}\frac{1}{7}$.

2. The angles of elevation of an aeroplane from two places one mile apart and from a point half way between them are found to be 60°, 30° and 45° respectively. Show that the height of the aeroplane is $440\sqrt{6}$ yards.

3. A building with ten storeys, each of equal height x ft., stands on one side of a wide street, and from a point on

the other side of the street directly opposite to the building, it is observed that the three uppermost storeys together and the two lowest storeys together subtend equal angles. Show that the width of the street is $x\sqrt{140}$ ft.

4. A two-storeyed building has the height of its lower storey 12 ft. and that of the upper storey 13 ft. Find at what distance the two storeys subtend equal angles to an observer's eye at a height 5 feet from the ground.

[*Ans.* $\sqrt{2135}$ ft.]

5. A vertical rod is erected in a horizontal rectangular field $ABCD$. The angular elevation of its top from A, B, C, D are α, β, γ, δ. Show that

$$\cot^2\alpha - \cot^2\beta = \cot^2\delta - \cot^2\gamma.$$

6. The angles of elevation of a bird flying in a horizontal straight line, from a fixed point at four successive observations are α, β, γ, δ, the observations being taken at equal intervals of time. Assuming that the speed of the bird is uniform, show that

$$\cot^2\alpha - \cot^2\delta = 3(\cot^2\beta - \cot^2\gamma).$$

7. A man on a hill observes that three towers on a horizontal plane subtend equal angles at his eye and that the angles of depression of their bases are α, β, γ. If a, b, c are the heights of the towers, prove that

$$\frac{\sin(\beta-\gamma)}{a\sin\alpha} + \frac{\sin(\gamma-\alpha)}{b\sin\beta} + \frac{\sin(\alpha-\beta)}{c\sin\gamma} = 0.$$

8. A gun is fired from a fort F at a distance d from a station O, and from two stations A and B in a straight line with O and distant a and b respectively from O, the intervals between seeing the flash and hearing the report are t and t'. Show that the velocity of sound is

$$\sqrt{\frac{(d^2-ab)(a-b)}{at'^2-bt^2}}.$$

9. A person observes the elevation of the top of a telegraph post which is E. S. E. of him to be 45°, and at noon, the extremity of its shadow is to the N. E. of him; if the length of the shadow be x, show that the height of the post is $x\sqrt{2}-\sqrt{2}$.

10. A straight tree on the horizontal ground leans towards the North ; at two points due South and distant a, b respectively from the foot, the angular elevations of the top of the tree are α and β. Show that the inclination of the tree to the horizon is

$$\tan^{-1}\left(\frac{a-b}{a\cot\beta - b\cot\alpha}\right).$$

11. An observer on a carriage moving with a speed V along a straight road observes in one position that two distant trees are in the same line with him which is inclined at an angle θ to the road. After a time t, he observes that the trees subtend their greatest angle ϕ ; show that the distance between the trees is

$$2Vt \sin\theta \sin\phi \,/\, (\cos\theta + \cos\phi).$$

12. A train travelling on one of two straight intersecting railways subtends at a certain station on the other line, angles α and β, when the front of the first carriage and the end of the last carriage reach the junction respectively. Show that the angle of intersection of the two lines is

$$\tan^{-1}\frac{2\sin\alpha\sin\beta}{\sin(\alpha\sim\beta)}.$$

13. Two vessels are sailing in parallel directions, and at one instant the bearing of one from the other is α° N. of E. After an hour's sailing the bearing of the first from the second is β° N. of E. and after another hour the bearing is γ° N. of E. Show that the vessels are sailing in a direction θ° N. of E., where

$$\sin(\alpha-\theta)\sin(\gamma-\beta) = \sin(\beta-\alpha)\sin(\gamma-\theta).$$

14. A rod of given length can turn in a vertical plane passing through the sun, one end being fixed on the ground ; if the longest shadow it can cast is $3\frac{1}{3}$ times the length of the rod, calculate the altitude of the sun, having given

$$\log 3 = \cdot 47712,\ L\cos 72^\circ\, 32'\, 30'' = 9\cdot 47712.$$

[*Ans.* $17^\circ\, 27'\, 30''$]

15. A ship sailing N. E. is, at a particular moment, in a line with two light-houses, one of which is situated 5 miles

due N. of the other. In 3 minutes and also in 21 minutes the light-houses are found to subtend a right angle at the ship. Prove that the ship is sailing at the rate of 10 miles an hour, and that the light-houses subtend their greatest angle at the ship at the end of $3\sqrt{7}$ minutes.

16. A parachute was observed in the N. E at the elevation 45°; ten minutes afterwards it was found to be due N. at an elevation $22\frac{1}{2}^\circ$. The parachute was descending at the rate of 6 miles per hour, and was all along drifted uniformly towards the west by the wind. Show that wind blows at the rate of 6 miles per hour.

17. A person wishing to determine the height of a distant temple observes the elevation of its top from a point on the horizontal ground through its base to be 30°. On walking a distance $80\sqrt{3}$ ft. in a certain direction, he finds the elevation of the top to be the same as before, and then on walking a distance 80 ft. at right angles to the former direction, the elevation is found to be 45°. Show that the height of the temple is 80 ft.

18. The shadow of a telegraph post is observed to be half the known height of the post, and sometime afterwards it is equal to the known height; how much will the sun have gone down in the interval, given

$$\log 2 = \cdot 30103,\ L \tan 63^\circ\ 26' = 10\cdot 3009994$$

$$\text{and diff. for } 1' = 3159. \qquad [\ \textit{Ans.}\ 18^\circ\ 26'\ 5\cdot 8''\ \text{nearly}\ ]$$

19. The side of a hill faces due S, and is inclined to the horizon at an angle α. A straight railway upon it is inclined at an angle β to the horizon; show that the bearing of the railway is

$$\cos^{-1}(\cot\alpha \tan\beta) \text{ E. of N.}$$

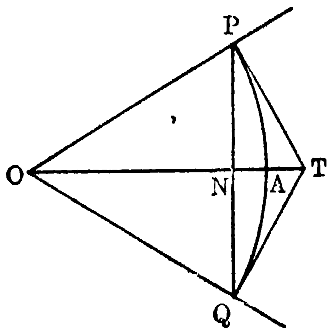
20. A spherical time-ball of diameter d at the top of a tower subtends an angle 2α at a point on the ground from which the elevation of its centre is θ; prove that the height of the centre of the ball above the ground is $\frac{1}{2}d \sin\theta \operatorname{cosec}\alpha$.

APPENDIX

1. *To prove that*

$$\sin\theta < \theta < \tan\theta$$

where θ is the circular measure of any positive acute angle.



Let AOP be a positive acute angle whose radian measure is θ, and let QOA be an equal angle on the other side of OA. With centre O and any radius, a circle is drawn cutting OP, OA, OQ at P, A, Q respectively. PQ is joined cutting OA at N. The triangles OPN and OQN are easily seen to be congruent, so that $PN = QN$ and PNQ is perpendicular to OA. The tangent PT to the circle at P cutting OA at T, $\angle OPT$ is a right angle. TQ being joined, the triangles OPT and OQT are easily proved to be congruent, so that $TP = TQ$.

The figure is thus symmetrical about OA.

Then, from the figure,

$$\sin\theta = \frac{PN}{OP} = \frac{1}{2}\cdot\frac{PQ}{OP}$$

$$\theta = \frac{\text{arc } PA}{OP} = \frac{1}{2}\,\frac{\text{arc } PAQ}{OP}$$

$$\tan\theta = \frac{PT}{OP} = \frac{1}{2}\cdot\frac{PT+QT}{OP}$$

Now, we may take it as axiomatic that the straight line PQ is less than the curved arc PAQ, and that the curved arc PAQ which always bends the same way, being within the triangle PTQ, is less than $PT + TQ$.

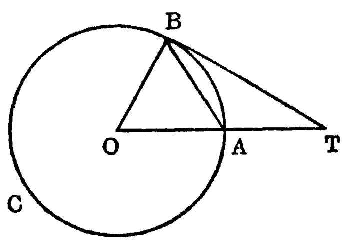
Hence, since $PQ <$ arc $PAQ < PT + QT$,

we have, on dividing throughout by $2OP$,

$$\sin\theta < \theta < \tan\theta.$$

Alternative method :

Let ABC be a circle whose centre is O and radius r.



Let $AOB = \theta$ radians.

Draw BT the tangent at B to meet OA produced at T; then $BT = r\tan\theta$.

We know that if the angle of a sector of a circle of radius r is θ radians, its area $= \frac{1}{2}r^2\theta$.

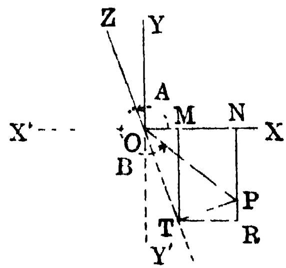
Now, from the figure it is obvious that if $0 < \theta < \frac{\pi}{2}$;

$$\triangle OAB < \text{sector } OAB < \triangle OBT.$$

$\therefore \quad \frac{1}{2}r^2\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r.r\tan\theta,$

i.e., $\quad \sin\theta < \theta < \tan\theta.$

2. *Formulæ for* **sin (A+B)** *and* **cos (A+B)** *where A and B are of any magnitude. (Generalisation of Art. 33).*



In Article 33, formulæ for $\sin(A+B)$ and $\cos(A+B)$ were deduced geometrically with a figure in which A and B were acute and $(A+B)$ less than 90°. We now prove them in a more general case.

Let a revolving line, starting from OX, trace out an angle $XOZ = A$ and further trace out an angle $ZOP = B$, so that the total angle traced out is $(A+B)$. From any point P on the final position of the revolving line, PN and PT are drawn perpendiculars to OX and OZ (produced if necessary, as in the above figure), and from T perpendiculars TM and TR are drawn on OX and PN (produced if necessary).

In the figure above, $\angle POT = B - 180°$, and since PN and PT are perpendiculars to OX and OZ respectively, $\angle TPR = \angle TON = 180° - \angle XOZ$, *i.e.*, $180° - A$.

In considering $\sin(A+B)$ and $\cos(A+B)$ from the triangle NOP, it is to be noted that PN is negative and ON and OP are positive.

If we consider only the positive magnitudes of the sides of the acute-angled triangles OTM, PTR and OPT, then PN with its proper sign may be written as $-(TM-PR)$, and ON with its proper sign may be written as $OM+TR$.

Now, from the figure,

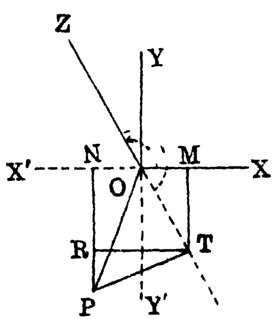
$$\sin(A+B) = \frac{PN}{OP} = -\frac{TM-PR}{OP}$$

$$= -\frac{TM}{OT}\frac{OT}{OP}+\frac{PR}{PT}\frac{PT}{OP}$$
$$= -\sin TOM \cos POT + \cos TPR \sin POT$$
$$= -\sin (180^\circ - A) \cos (B - 180^\circ)$$
$$\qquad + \cos (180^\circ - A) \sin (B - 180^\circ)$$
$$= -\sin A\,(-\cos B) + (-\cos A)(-\sin B)$$
$$= \sin A \cos B + \cos A \sin B.$$

Again,

$$\cos (A + B) = \frac{ON}{OP} = \frac{OM + RT}{OP}$$
$$= \frac{OM}{OT}\frac{OT}{OP} + \frac{RT}{PT}\frac{PT}{OP}$$
$$= \cos TOM \cos POT + \sin TPR \sin POT$$
$$= \cos (180^\circ - A) \cos (B - 180^\circ)$$
$$\qquad + \sin (180^\circ - A) \sin (B - 180^\circ)$$
$$= (-\cos A)(-\cos B) + \sin A\,(-\sin B)$$
$$= \cos A \cos B - \sin A \sin B.$$

3. *Formulae for* **sin (A − B)** *and* **cos (A − B)** *in a more general case* (*Generalisation of Art 34*)



Here, XOZ measured counter-clockwise is A and ZOP measured clockwise has magnitude B so that XOP measured

clockwise is $A - B$. From P, PN and PT are drawn perpendiculars on OX and OZ (produced in this figure), and from T, TM and TR are drawn perpendiculars on OX and PN.

In the present figure, magnitudes of the acute angles TOM and POT are $180° - A$ and $B - 180°$ respectively, and noting that $PNOT$ is a cyclic quadrilateral ($\angle^s$ N and T being right angles), $\angle RPT = \angle TOM = 180° - A$ in magnitude.

Now, we see that in considering $\sin(A-B)$ and $\cos(A-B)$ from the triangle NOP, PN and ON are of negative sign.

Hence,

$$\sin(A-B) = \frac{PN}{OP}$$

$$= -\frac{MT + PR}{OP},$$

where the magnitudes of MT, PR, etc. only are considered,

$$= -\frac{MT}{OT}\cdot\frac{OT}{OP} - \frac{PR}{PT}\cdot\frac{PT}{OP}$$

$$= -\sin TOM \cos POT - \cos RPT \sin POT$$

$$= -\sin(180° - A)\cos(B - 180°)$$
$$\quad - \cos(180° - A)\sin(B - 180°)$$

$$= -\sin A(-\cos B) - (-\cos A)(-\sin B)$$

$$= \sin A \cos B - \cos A \sin B.$$

Similarly,

$$\cos(A-B) = \frac{ON}{OP} \qquad [\text{ where } ON \text{ is taken with proper sign }]$$

$$= -\frac{RT - OM}{OP} \qquad [\text{ where magnitudes only of } RT,\ OM \text{ etc. are considered }]$$

$$= -\frac{RT}{PT}\cdot\frac{PT}{OP} + \frac{OM}{OT}\cdot\frac{OT}{OP}$$

$$= -\sin RPT \sin POT + \cos TOM \cos POT$$

$$= -\sin(180° - A)\sin(B - 180°)$$
$$\quad + \cos(180° - A)\cos(B - 180°)$$

$$= -\sin A\,(-\sin B) + (-\cos A)(-\cos B)$$
$$= \cos A \cos B + \sin A \sin B.$$

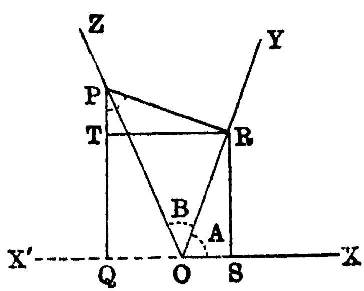
4. A few particular cases of sin (A ± B), cos (A ± B).

Case I. In the case A and B are both acute and $(A+B) > 90°$.

Construction same as in Art. 33. Here, Q, the foot of the perpendicular will fall on XO produced.

$$\angle TPR = 90° - \angle TRP = \angle TRO - \angle ROS = A.$$

$$\sin(A+B) = \sin XOP$$
$$= \frac{PQ}{OP} = \frac{QT+TP}{OP}$$
$$= \frac{RS+PT}{OP} = \frac{RS}{OP} + \frac{PT}{OP}$$
$$= \frac{RS}{OR}\cdot\frac{OR}{OP} + \frac{PT}{PR}\cdot\frac{PR}{OP}$$
$$= \sin A \cos B + \cos TPR \sin B$$
$$= \sin A \cos B + \cos A \sin B.$$



$$\cos(A+B) = \cos XOP = -\frac{OQ}{OP},$$ [Magnitude of OQ being considered]

$$= -\frac{SQ-SO}{OP} = \frac{OS}{OP} - \frac{SQ}{OP} = \frac{OS}{OP} - \frac{TR}{OP}$$

$$=\frac{OS}{OR}\cdot\frac{OR}{OP}-\frac{TR}{PR}\cdot\frac{PR}{OP}$$
$$=\cos A\cos B-\sin TPR\sin B$$
$$=\cos A\cos B-\sin A\sin B$$

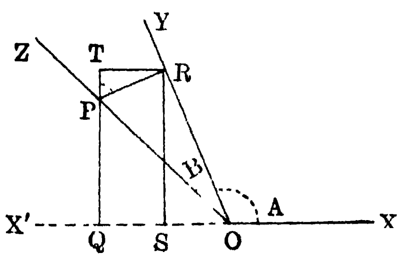
Case II. In the case A is obtuse and B is acute and $(A+B)<180°$.

Construction same as in Art. 33.

Here, $\angle TPR=180°-\angle RPQ=\angle ROQ=180°-A$.

$\therefore\quad \sin TPR=\sin A,\ \cos TPR=-\cos A.$

$$\sin(A+B)=\sin XOP=\frac{PQ}{OP}=\frac{QT-PT}{OP}=\frac{RS-PT}{OP}$$
$$=\frac{RS}{OP}-\frac{PT}{OP}=\frac{RS}{OR}\cdot\frac{OR}{OP}-\frac{PT}{PR}\cdot\frac{PR}{OP}$$
$$=\sin A\cos B-\cos TPR\sin B$$
$$=\sin A\cos B+\cos A\sin B$$



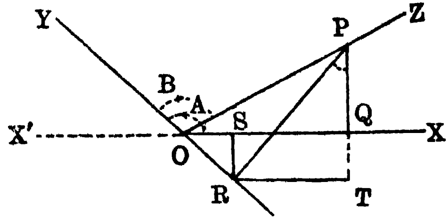
$$\cos(A+B)=\cos XOP=-\frac{OQ}{OP}$$ [Magnitude of OQ being considered]

$$=-\frac{OS+SQ}{OP}=-\frac{OS}{OP}-\frac{SQ}{OP}$$
$$=-\frac{OS}{OR}\cdot\frac{OR}{OP}-\frac{TR}{PR}\cdot\frac{PR}{OP}$$
$$=\cos A\cos B-\sin TPR\sin B$$
$$=\cos A\cos B-\sin A\sin B.$$

Case III. In the case A and B are both obtuse and $(A-B)$ is acute.

Construction same as in Art. 34.

Here, $\angle TPR = \angle ROS = 180° - A$.



$$\sin(A-B) = \sin POQ$$
$$= \frac{PQ}{OP} = \frac{PT-RS}{OP}$$
$$= \frac{PT}{OP} - \frac{RS}{OP} = \frac{PT}{PR}\cdot\frac{PR}{OP} - \frac{RS}{OR}\cdot\frac{OR}{OP}$$
$$= \cos TPR \sin POR - \sin ROS \cos POR$$
$$= \cos(180° - A)\sin(180° - B) - \sin(180° - A)\cos(180° - B)$$
$$= -\cos A \sin B - \sin A(-\cos B)$$
$$= \sin A \cos B - \cos A \sin B.$$

$$\cos(A-B) = \cos POQ$$
$$= \frac{OQ}{OP} = \frac{OS+SQ}{OP} = \frac{OS+RT}{OP} = \frac{OS}{OP} + \frac{RT}{OP}$$
$$= \frac{OS}{OR}\cdot\frac{OR}{OP} + \frac{RT}{PR}\cdot\frac{PR}{OP}$$
$$= \cos ROS.\cos POR + \sin TPR.\sin POR$$
$$= \cos(180° - A)\cos(180° - B) + \sin(180° - A)\sin(180° - B).$$
$$= \cos A \cos B + \sin A \sin B.$$

5. Note of Art. 90.

Let us denote the formulæ of Arts. 82, 83, 84 by (I), (II), (III). We have seen in Art. 90, that (II) can be deduced from (III). We shall now show how any one of these can be deduced from any other of the rest.

To deduce (I) from (III).

Substituting the value of b from the second relation of Art. 84 in the first,

$$a=(c \cos A+a \cos C) \cos C+c \cos B.$$

$$\therefore \quad a\,(1-\cos^2 C)-c\,(\cos A \cos C+\cos B)$$
$$=c\,\{\cos A \cos C-\cos (A+C)\}$$
$$[\because A+B+C=\pi]$$

$$\therefore \quad a \sin^2 C=c \sin A \sin C. \quad \therefore \quad a/\sin A=c/\sin C.$$

Similarly, substituting the value of c in the first relation, we get

$$a/\sin A=b/\sin B. \quad \text{Hence, etc.}$$

To deduce (II) and (III) from (I).

(i) Putting each of the ratios of Art. 82 equal to k, we get

$$a=k \,.\, \sin A\;;\; b=k \,.\, \sin B\;;\; c=k \,.\, \sin C.$$

$$\therefore \quad \frac{b^2+c^2-a^2}{2bc}=\frac{k^2\,(\sin^2 B+\sin^2 C-\sin^2 A)}{k^2 \,.\, 2 \sin B \sin C}$$

$$=\frac{\sin^2 B+\sin (C+A) \sin (C-A)}{2 \sin B \sin C}$$

$$=\frac{\sin B\,\{\sin B+\sin (C-A)\}}{2 \sin B \sin C}$$

$$[\because \sin (C+A)=\sin (\pi-B)=\sin B]$$

$$=\frac{\sin B\,\{\sin (C+A)+\sin (C-A)\}}{2 \sin B \sin C}$$

$$=\frac{2 \sin B \sin C \cos A}{2 \sin B \sin C}=\cos A.$$

(ii) $b \cos C + c \cos B = k (\sin B \cos C + \sin C \cos B)$

$$= k \sin (B + C) = k \sin A$$

$$= a. \qquad [\because \; A+B+C=\pi]$$

To deduce (I) and (III) from (II)

(i) $\sin^2 A = 1 - \cos^2 A$

$$= 1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2 = \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4b^2c^2}$$

$$= \frac{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)}{4b^2c^2}$$

$$= \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$$

$$= \frac{K}{4b^2c^2} \text{ say.}$$

$$\therefore \frac{\sin^2 A}{a^2} = \frac{K}{4a^2b^2c^2};$$

similarly, $\dfrac{\sin^2 B}{b^2}$ and $\dfrac{\sin^2 C}{c^2}$ each $= \dfrac{K}{4a^2b^2c^2}$.

$$\therefore \frac{\sin^2 A}{a^2} = \frac{\sin^2 B}{b^2} = \frac{\sin^2 C}{c^2}; \text{ hence etc.}$$

(ii) Adding 2nd and 3rd relations of the formulæ of Art. 83, we get

$$b^2 + c^2 = b^2 + c^2 + 2a^2 - 2ca \cos B - 2ab \cos C.$$

Now, transposing and dividing by $2a$, we get

$$a = b \cos C + c \cos B \text{ ; etc.}$$

ANSWERS

Examples I. [Pages 11-14]

1. (i) first quadrant. (ii) third quadrant.

(iii) second quadrant. (iv) fourth quadrant.

2. (i) $61^g\ 34'\ 44''\cdot 4$. (ii) $175°\ 49'\ 1''\cdot 776$.

3. (i) $\cdot 253775\pi$. (ii) $\frac{33}{320}\pi$.

4. $82°\ 30'$; $91^g\ 66'\ 66''\cdot 6$; $\frac{11}{24}\pi$. **5.** $\alpha : \beta = 5\pi : 24$.

6. $\frac{1}{2}\left(1-\frac{\pi}{180}\right)$. **7.** 6° and 9°. **8.** $\frac{1}{90}\left(D+\frac{M}{60}\right)-\frac{1}{100}\left(G+\frac{m}{100}\right)$.

9. [illegible] nearly. **10.** 20° and 30°. **11.** $20^g, 40^g, 80^g$.

13. 27°, 9°, 18°. **14.** (i) At $28\frac{4}{11}$ min. and 48 min. past 7.

(ii) At 7-10. **15.** 20°, 60°, 100°. **16.** $\frac{\pi}{7}, \frac{2\pi}{7}, \frac{4\pi}{7}; \frac{\pi}{21}, \frac{4\pi}{21}, \frac{16\pi}{21}$.

17. 45°, 60°, 120°, 135°. **18.** 9.

19. mx and nx where $x=\frac{2\,(10pm-9qn)}{mn\,(10p-9q)}$. **20.** $\frac{3}{4}$.

21. 3 and 6. **22.** 51·41 miles per hour (nearly).

23. 66444 miles per hour (nearly) ; 431445 miles (nearly).

24. 76·8 ft. (nearly). **25.** 3959 miles (nearly). **26.** 33 ft. **27.** 360 yds.

Examples II. [Pages 24-26]

25. $(\sin\theta-\cos\theta)^2$. **26.** $\frac{1}{\tan^4\theta}-\tan^4\theta$. **31.** $\frac{a^2-b^2}{a^2+b^2}$.

33. $\pm\frac{\sqrt{\sec^2\alpha-1}}{\sec\alpha}$; $\pm\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$. **34.** $\frac{16}{65}$. **36.** $\frac{1}{2}$. **37.** 1 or $\frac{1}{2}$.

39. $\frac{a^2-b^2}{2ab}$; $\frac{a^2+b^2}{a^2-b^2}$. **43.** (i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. (ii) $xy=c^2$

(iii) $(bc'-b'c)^2+(ca'-c'a)^2=(ab'-a'b)^2$

(iv) $(a'b-b'c)(ab'-bc')=(aa'-cc')^2$.

Examples III. [Pages 35-36]

7. $\frac{\sqrt{3}}{2}$. **8.** (i) 60°. (ii) 45°. (iii) 30° (There is another angle which is not one of the standard angles).
(iv) 45°. (v) 30°. (vi) 30°. (vii) 30°.

9. $\theta = 52\frac{1}{2}°$, $\phi = 7\frac{1}{2}°$. **10.** $\alpha = 50°$, $\beta = 10°$.

11. $A = 22\frac{1}{2}°$, $B = 67\frac{1}{2}°$, $C = 45°$. **12.** (i) $-\frac{15}{3}$. (ii) 1.

Examples IV. [Pages 49-51]

1. $\frac{1}{2}$; $-\frac{1}{\sqrt{3}}$; $\frac{2}{\sqrt{3}}$; -1. **2.** $-\frac{1}{\sqrt{2}}$; $-\frac{2}{\sqrt{3}}$; $-\frac{1}{\sqrt{3}}$; $\frac{\sqrt{3}}{2}$. **3.** 0.

4. $\frac{\sqrt{3}}{2}$. **5.** (i) 1. (ii) ± 2, $\pm\frac{2}{\sqrt{3}}$. **10.** $\tan^3\theta$; 1. **12.** (i) 2.
(ii) 1. (iii) $\sin x$ or 0 according as n is odd or even. **13.** $\frac{61}{28}$.

14. $\frac{\sqrt{46}}{8}$. **15.** (i) cot 26°. (ii) cos 25°. (iii) cosec 39°. (iv) $\cos\frac{\pi}{9}$.

16. (i) 300°. (ii) 480°. **17.** (i) 60°. (ii) 120°, 240°.
(iii) 30°, 150°, 210°, 330°. (iv) 30°, 150°. (v) 30°, 135°, 150°, 315°.

Examples V. [Pages 56-59]

1. $100\sqrt{3}$ ft. **2.** 2·89... miles ; $2\frac{1}{2}$ miles **3.** $20\sqrt{3}$ ft. ; 120 ft.
4. $20\sqrt{3}$ ft. ; 20 ft. **5.** $30\sqrt{2}$ ft. **6.** $400(\sqrt{3}+1)$ yds.
7. $40\sqrt{3}$ ft. **8.** $\frac{1}{2}(3 \pm \sqrt{3})$ miles. **9.** 22·3 miles nearly.
10. 94·64 ft. nearly **11.** 47·32 ft. nearly. **12.** 60 miles per hour.
13. $50\sqrt{6}$ ft. **14.** $40\sqrt{6}$ ft. ; $40\sqrt{2}(\sqrt{7}+1)$ ft.
15. $\frac{1}{2}(\sqrt{3}+1)$ miles. **16.** $5\sqrt{13}$ miles.
17. 211·6... ft. ; 91·6... ft. **18.** 5·25... miles per hour.
19. 367·38 ft. **20.** $\frac{1}{4}\sqrt{6}(\sqrt{5}+1)$.
22. 2 miles. **23.** 13·66 ft.

Examples VI. [Pages 68-70]

21. $\sin A \cos B \cos C - \sin B \cos C \cos A + \sin C \cos A \cos B + \sin A \sin B \sin C$;

$$\frac{\tan A - \tan B - \tan C - \tan A \tan B \tan C}{1 + \tan A \tan B + \tan A \tan C - \tan B \tan C}$$

22. $$\frac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot B \cot C + \cot C \cot A + \cot A \cot B - 1}.$$

Examples VII. [Pages 79-81]

27. a.

Examples IX. [Pages 86-87]

16. $\dfrac{b^2-a^2}{b^2+a^2}$. **17.** (i) $2\sin\frac{1}{2}A=\sqrt{1+\sin A}+\sqrt{1-\sin A}$.

(iii) No ; $2\sin\frac{1}{2}\theta=\sqrt{1+\sin\theta}+\sqrt{1-\sin\theta}$.

Examples XI. [Pages 110-111]

1. $n\pi\pm\dfrac{\pi}{4}$, i.e., $(2k+1)\dfrac{\pi}{4}$. **2.** (i) $n\pi\pm\dfrac{\pi}{4}$. (ii) $n\pi\pm\dfrac{\pi}{3}$.

3. $2n\pi\pm\dfrac{\pi}{3}$, $(2k+1)\pi$. **4.** $\dfrac{n\pi}{2}+(-1)^n\dfrac{\pi}{12}$.

5. $n\pi+\dfrac{\pi}{4}$, or, $n\pi+(-1)^n\dfrac{\pi}{6}$. **6.** $\dfrac{n\pi}{3}$, or, $n\pi\pm\dfrac{\pi}{6}$.

7. $\dfrac{r\pi}{m+(-1)^r n}$. **8.** $(2n+1)\dfrac{\pi}{2}$, or, $(2n+1)\dfrac{\pi}{4}$, or, $(2n+1)\dfrac{\pi}{8}$.

9. $n\pi-\dfrac{\pi}{4}$, or, $\dfrac{n\pi}{2}+(-1)^n\dfrac{a}{2}$, where $\sin a=\dfrac{\sqrt{5}-1}{2}$. **10.** $\frac{1}{4}\pi$.

11. $n\pi+\dfrac{\pi}{4}$. **12.** $(4n+1)\dfrac{\pi}{8}$. **13.** $2n\pi+\dfrac{5\pi}{12}$, or, $2n\pi-\dfrac{\pi}{12}$.

14. $(2n+1)\dfrac{\pi}{4}$, or, $n\pi\pm\dfrac{\pi}{6}$. **15.** $2n\pi+\dfrac{\pi}{2}$, or, $2n\pi-\beta$, where β is a positive acute angle whose sine is $\frac{3}{5}$. **16.** $\frac{1}{6}n\pi$. **17.** $n\pi\pm\frac{1}{6}\pi$.

18. $(4n+1)\dfrac{\pi}{12}$. **19.** $2n\pi+\frac{7}{12}\pi$, or, $2n\pi+\frac{1}{12}\pi$. **20.** $-\frac{5}{6}\pi$, $-\frac{1}{6}\pi$, $\frac{1}{2}\pi$, $\frac{11}{6}\pi$.

21. $\frac{1}{2}(n\pi+a)$, where $\tan a=2$. **22.** $2n\pi$. **23.** $2n\pi$, $\frac{1}{6}(4n+1)\pi$.

24. 90°, 450°, 810°. **25.** $\frac{1}{4}\pi$, $\frac{1}{4}\pi$. **27.** (i) $\frac{1}{2}n\pi+\frac{1}{4}\pi$; $2n\pi\pm\frac{2}{3}\pi$.

(ii) $0, \pm\dfrac{\pi}{12}, \pm\dfrac{\pi}{6}, \pm\dfrac{\pi}{4}$. (iii) $\dfrac{n\pi}{3}$, $n\pi\pm\tan^{-1}\dfrac{1}{\sqrt{2}}$. (iv) $2n\pi-a$, $\dfrac{4n-1}{2}\pi+a$.

(v) $2n\pi$, or, $2n\pi-\frac{1}{2}\pi$. (vi) $(2n+1)\dfrac{\pi}{2}$, $\dfrac{4n+1}{14}\pi$, $\dfrac{4n-1}{6}\pi$.

(vii) $n\pi+\dfrac{a}{2}$; $(2n+1)\dfrac{\pi}{6}-\dfrac{a}{6}$. **28.** $n\pi+(-1)^n 21^\circ 48'-68^\circ 12'$.

29. (i) $a=\beta=\frac{1}{4}\pi$; or, $a=\frac{3}{4}\pi$, $\beta=-\frac{1}{4}\pi$.
(ii) $a=\frac{1}{4}\pi$, $\beta=\frac{1}{12}\pi$; or, $a=\frac{11}{12}\pi$, $\beta=\frac{3}{4}\pi$;
or, $a=\frac{3}{4}\pi$, $\beta=\frac{5}{12}\pi$; or, $a=\frac{7}{12}\pi$, $\beta=-\frac{1}{4}\pi$.

Examples XII. [Pages 119-121]

22. (i) 1. (ii) ∞. (iii) $\frac{x+y}{1-xy}$. **23.** $y=\frac{4x\,(1-x^2)}{1-6x^2+x^4}$.

24. $(x-y)(1+yz)=(y-z)(1+xy)$. **25.** (i) $\frac{1}{4}$, or, -8. (ii) $\frac{a-b}{1+ab}$.

(iii) $\pm\frac{\sqrt{5}}{3}$. (iv) $\pm\frac{1}{\sqrt{3}}$. (v) $\frac{4}{3}$, or, $-\frac{8}{5}$. (vi) $\pm\frac{1}{14}\sqrt{21}$.

(vii) 0, or, $\frac{1}{2}$. (viii) 0, $\pm\frac{1}{2}$. (ix) $2-\sqrt{3}$. (x) $\frac{6+\sqrt{6}}{3}$.

Miscellaneous Examples I. [Pages 122-123]

2. $\pm\sqrt{\frac{b^2-c^2}{a^2-c^2}}$. **19.** $a^2+b^2=2\,(1+c)$.

Examples XIII(a). [Pages 135-137]

1. (i) 6. (ii) -3. **2.** -2. **5.** $\frac{n}{n-1}$. **9.** (i) 1. (ii) $1\frac{1}{2}$.

10. $\bar{1}$·1173942, ·3861209. **13.** 2·425805. **14.** ·41369.

15. (i) $\bar{1}$·8969092. (ii) ·898665. **16.** 39·879.

17. (i) 13. (ii) 6. (iii) 25. **18.** (i) 24. (ii) 4. (iii) 79.

19. (i) $\frac{\log 2}{\log 3}$, *i.e.*, ·63...... (ii) $4+\frac{\log 7}{\log 3}$, *i.e.*, 5·77...

(iii) $\frac{2\log 7-3\log 3}{6\log 5-\log 7-2\log 3}$, *i.e.* ·108...

(iv) $x=\frac{\log 3}{\log 3-\log 2}=2·71$ nearly, $y=\frac{\log 2}{\log 3-\log 2}=1·71$ nearly.

(v) $\frac{2b\,(2a-b)}{5ab+3ac-2b^2-bc}$ and $\frac{2ab}{5ab+3ac-2b^2-bc}$,

where $a=\log 2$, $b=\log 3$, $c=\log 7$.

20. (i) $\log x=\frac{a+3b}{5}$, $\log y=\frac{a-2b}{5}$.

Examples XIII(b). [Pages 142-144]

1. 3·2766077. **2.** $\bar{1}$·3686646. **3.** 37·6018. **4.** ·7400827.

5. ·8455104 ; 32° 16′ 21″. **6.** ·7928863.

7. 9·8440554, 10·1559446. **8.** 36° 24′ 36″.

9. 53° 13′ 55″. **10.** 9·6198509 ; 22° 36′ 28″.

12. 10·0957589. **13.** 9·9147334. **14.** 9·8718486.

16. $\theta=50°\ 7'\ 48''$ nearly. **17.** ·2394.

Examples XIV(a). [Pages 157-160]

23. 120°. **24.** $A=60°$. **29.** $A=90°$, $B=30°$, $C=60°$.

39. $\sqrt{\frac{y}{z}+\frac{z}{x}+\frac{x}{y}}$. **40.** 84.

Examples XIV(b). [Pages 166-168]

15. $r=4$; $R=8\frac{1}{8}$.

Examples XV(a). [Pages 172-173]

1. 35° 5′ 49″. **2.** 102° 1′ 28″. **3.** 58° 59′ 33″.

4. 104° 30′; 46° 36′; 28° 54′. **5.** (i) 88° 59′ 40·9″.

(ii) 78° 27′ 46·86″. **6.** (i) 48° 11′ 23″; 58° 24′ 43″; 73° 23′ 54″.

(ii) 132° 34′ 24″. **7.** $A=120°$, $B=45°$, $C=15°$.

8. $A=45°$, $B=30°$, $C=105°$. **9.** $A=60°$, $B=38° \ 11'$, $C=81° \ 49'$.

10. $A=105°$, $B=45°$, $C=30°$. **11.** $(\sqrt{3}+1) : \sqrt{6} : (\sqrt{3}-1)$.

13. $\sqrt{5}+1 : \sqrt{5}-1$. **14.** $3 : 4 : 5$.

Examples XV(b). [Pages 176-178]

1. $B=38° \ 12' \ 48''$, $C=21° \ 47' \ 12''$.

2. $B=56° \ 19' \ 46{\cdot}3''$, $C=63° \ 40' \ 13{\cdot}7''$.

3. $A=117° \ 38' \ 45''$, $B=27° \ 38' \ 45''$.

4. $A=94° \ 42' \ 54''$, $B=25° \ 17' \ 6''$.

5. $B=71° \ 44' \ 29{\cdot}5''$, $C=48° \ 15' \ 30{\cdot}5''$.

6. (i) 70° 53′ 36″; 49° 6′ 14″. (ii) 74° 13′ 50″, 35° 16′ 10″.

(iii) $A=64° \ 21'$, $B=77° \ 25'$, $c=27{\cdot}39$.

7. (i) $B=78° \ 17' \ 39{\cdot}6''$, $C=49° \ 36' \ 20{\cdot}4''$.

(ii) 116° 33′ 54″; 25° 33′ 54″.

8. $A=B=75°$, $C=30°$, $b=2\sqrt{6}$. **9.** $\sqrt{6}$, 15°, 105°.

10. (i) $A=45°$, $B=75°$, $C=\sqrt{6}$. (ii) $A=30°$, $B=90°$.

11. 27·0375. **12.** 172·6436 ft. **13.** 79·063.

14. (i) $A=31° \ 20'$, $b=185$, $c=192$.

(ii) $b=18{\cdot}46$, $c=37{\cdot}16$, $C=70° \ 30'$. (iii) $b=113{\cdot}9$, $c=117{\cdot}2$.

15. $C=75°$, $a=c=2\sqrt{3}+2$. **16.** $C=105°$, $a=\sqrt{2}$, $c=\sqrt{3}+1$.

17. 72°, 72°, 36°; each side $=\sqrt{5}+1$. **18.** 8, 1.

Examples XV(c). [Pages 184-185]

1. (i) One solution. (ii) Ambiguous; two solutions.

(iii) No solution. (iv) One solution (right-angled triangle).

2. (i) $C=75°$, $A=60°$, $a=\sqrt{6}$, or, $C=135°$, $A=30°$, $a=\sqrt{2}$ (ii) 60°, or, 120°.

3. $A=45°$, $C=75°$, $c=\sqrt{3}+1$. (no ambiguity). **4.** Impossible.

8. $C=58°\ 56'\ 56{\cdot}3''$, $A=87°\ 48'\ 3{\cdot}7''$ or, $C=121°\ 3'\ 3{\cdot}7''$, $A=25°\ 41'\ 56{\cdot}3''$

9. $B=34°\ 27'$, $C=100°\ 33'$.

10. $A=5°\ 44'\ 21''$. **11.** $A=33°\ 39'\ 34''$, $B=86°\ 20'\ 26''$.

12. $A=80°\ 36'$, $C=64°\ 14'$, or, $A=29°\ 4'$, $C=115°\ 46'$.

Miscellaneous Examples II. [Pages 186-188]

11. 4, 5, 6 **14.** $B=44°\ 25'\ 39''$, or, $135°\ 34'\ 4''$.

21. $\frac{1}{3}\{n\pi+\frac{1}{2}\pi-(a+b+c)\}$ **24.** $\frac{1}{3}(n\pi+\frac{1}{4}\pi)$.

Examples XVI. [Pages 213-214]

4. $\theta=\frac{1}{4}\pi$. **5.** $x=38°\ 10'$ nearly. **6** $\frac{1}{4}\pi$. **7.** $-{\cdot}37$ nearly.

8. (i) $x=0$. (ii) $46°\ 25'$ (nearly) and $90°$. (iii) $22\frac{1}{2}°$ and $112\frac{1}{2}°$.
(iv) $\frac{3}{8}\pi$. (v) $14°$ nearly. (vi) 1·19, 2·72, 4·92.
(vii) 1·16, 3·28, 4·95. (viii) $\pm$ ·82. (ix) ·64.

CALCUTTA UNIVERSITY QUESTIONS PAPERS

Pre-University Papers

1961

1. (*a*) Shew, geometrically, that if A and B are positive acute angles, and $A > B$, then

$$\cos(A-B)=\cos A\cos B+\sin A\sin B.$$

(*b*) If A, B, C are the angles of a triangle, shew that

$$\sin\frac{A}{2}+\cos\frac{B-C}{2}=2\cos\frac{B}{2}\cos\frac{C}{2}.$$

2. (*a*) Shew that

$$\frac{1+\sin\theta}{1-\sin\theta}=\tan^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

(*b*) Shew that

$$\tan^{-1}\tfrac{1}{3}+\tan^{-1}\tfrac{1}{5}+\tan^{-1}\tfrac{1}{7}+\tan^{-1}\tfrac{1}{8}=\tfrac{1}{4}\pi.$$

3. (*a*) Prove that with the usual notations

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}\,;\quad \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}.$$

(*b*) The sides of a triangle are 8 cms., 15 cms., 17 cms. Find its greatest angle.

4. (*a*) Find the value of $\sin 18°$.

(*b*) Draw a neat graph of $\sin x$ in the region $-\pi \le x \le \pi$.

1962

1. (*a*) Shew, geometrically that, if A and B are positive acute angles and $A+B < 90°$, then

$$\sin(A+B)=\sin A\cos B+\cos A\sin B.$$

(*b*) Shew that

$$\cos A+\cos B+\cos C+\cos(A+B+C)$$
$$=4\cos\frac{A+B}{2}\cos\frac{B+C}{2}\cos\frac{C+A}{2}.$$

2. (*a*) Solve and find a general value for x, where

$$\cos x + \sqrt{3} \sin x = 1.$$

(*b*) Shew that

$$\tan^{-1} x + \cot^{-1} y = \tan^{-1} \frac{xy+1}{y-x}.$$

3. (*a*) If a, b, c are the sides and A, B, C the angles opposite to the sides a, b, c respectively, in a triangle ABC, shew that

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

(*b*) A triangular courtyard has two of its sides of length 32 and 48 metres respectively and the angle included between those sides is $64° 36'$. Calculate the angles at the remaining vertices of the courtyard, having given

$$\log_{10} 2 = \cdot 30103\ ;\ L \cot 32° 18' = 10 \cdot 19916\ ;$$
$$L \tan 17° 33' = 9 \cdot 50004\ ;\ L \tan 17° 34' = 9 \cdot 50048.$$

4. (*a*) The shadow cast by a telegraph post is 6 metres longer when the sun's altitude is $30°$ then when it is $45°$; shew that the height of the post is $3 (1 + \sqrt{3})$ metres.

(*b*) Draw a neat graph of $\cos x$ in the region $-\pi \leq x \leq \pi$.

BOARD OF SECONDARY EDUCATION, WEST BENGAL

Higher Secondary Examination Papers

1960

1. (*a*) Prove that the radian is a constant angle. Find its value in degrees, minutes etc. [$\pi = \frac{22}{7}$]

(*b*) The angles of a triangle are in Arithmetical Progression and the number of degrees in the least is to the number of radians in the greatest as 60 to π. Find the angles in degrees.

2. (*a*) If A, B, $A+B$ are all acute angles, prove (geometrically) that

$$\cos (A+B) = \cos A \cos B - \sin A \sin B.$$

(*b*) Find the value of

$$\sin^2 60° + \cos^2 150° + \tan^2 120° + \cos 180° - \tan 135°.$$

3. (*a*) Find the values of θ between 0° and 360° which satisfy the equation $2 \sin^2\theta + 3 \cos \theta = 0$.

(*b*) If $A+B=90°$, prove that

$$\frac{\cos 2B - \cos 2A}{\sin 2A} = \tan A - \tan B.$$

4. (*a*) In a triangle ABC, prove that $a = b \cos C + c \cos B$.

(*b*) In a triangle, the angles are to one another as 1 : 2 : 3; prove that the corresponding sides are as $1 : \sqrt{3} : 2$.

5. Two vertical pillars the height of one of which is double that of the other are at a distance of 150 ft. from each other. At a point between the pillars and on the line joining their feet the angular elevations of the tops of the taller and the shorter pillar are found to be 60° and 30° respectively. Find the heights of the pillars and the position of the point.

6. Draw the graph of $\sin x$ between the values $x = -\pi$ and $x = \pi$ and find, from the graph, the value of sin 120°.

1960 (Compartmental)

1. (*a*) The difference between the two acute angles of a right-angled triangle is $\frac{2}{5}\pi$ radians; express these angles in degrees.

(*b*) If s is the length of the arc of a circle whose radius is r and θ is the radian measure of the angle at the centre, standing on the arc, prove that $\theta = s/r$.

2. (*a*) If A and B are both acute angles and A is greater than B, prove (geometrically) that

$$\sin (A-B) = \sin A \cos B - \cos A \sin B.$$

(b) If $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, where A and B are acute angles, find the value of

$$\frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

3. (a) Find the values of θ between $0°$ and $360°$ which satisfy the equation

$$\sin^2 \theta - 2 \cos \theta + \tfrac{1}{4} = 0.$$

(b) If $A + B + C = 180°$, prove that

$$\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.$$

4. In a triangle ABC, prove that

(i) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$. (ii) $a \cos \dfrac{B-C}{2} = (b+c) \sin \dfrac{A}{2}$.

5. The upper part of a straight tree broken over by the wind, but not completely separated, makes an angle of $30°$ with the ground, and the distance from the root to the point where the top of the tree touches the ground is 50 feet. What was the height of the tree?

6. Draw the graph of $\cos x$ between the values of $x = -\pi$ and $x = \pi$ and read off from the graph, the value of $\cos 150°$.

1961

1. (a) The radius of a circle is 10 cm.; find the angle, in degrees and minutes, subtended at its centre by an arc 6 cm. in length. [$\pi = \frac{22}{7}$]

(b) The angles of a triangle are in Arithmetical Progression. If the number of degrees in the greatest angle is the same as the number of grades in the least, find the angles in degrees.

2. (a) If A, B and $A - B$ are positive acute angles, prove geometrically that

$$\sin (A - B) = \sin A \cos B - \cos A \sin B.$$

(b) Find the value of

$$\sin 330° + \tan 45° - 4 \sin^2 120° + 2 \cos^2 135° + \sec^2 180°.$$

3. (a) Find the values of θ between $0°$ and $360°$ which satisfy the equation

$$\sqrt{3} \sin \theta + \cos \theta = 1.$$

(b) If $A + B + C = 180°$, prove that

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

4. In a triangle ABC, prove that

(a) $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

(b) $a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0$.

5. On a straight coast there are three objects A, B and C such that $AB = BC = 4$ miles. A steamer approaches B in a line perpendicular to the coast and at a certain point AC is found to subtend an angle of 60°; after sailing in the same direction for ten minutes, AC is found to subtend an angle of 120°; find the rate at which the steamer is going.

6. Draw the graph of $\sin x$ between the values of $x = 0^\circ$ and $x = 360^\circ$ and read off from the graph, the value of $\sin 240^\circ$.

1961 (Compartmental)

1. (*a*) Define a radian. Taking $\pi = 3{\cdot}1416$, show that a radian contains 206265 seconds approximately.

(*b*) One angle of a triangle is $\frac{2}{3}x$ grades and another is $\frac{3}{2}x$ degrees, whilst the third is $\frac{\pi x}{75}$ radians; express them all in degrees.

2. (*a*) If A, B and $A-B$ are all positive acute angles, prove geometrically that

$$\cos (A-B) = \cos A \cos B + \sin A \sin B.$$

(*b*) Find the value of

$$\frac{2 \tan^2 30^\circ}{1-\tan^2 30^\circ} + (\sec^2 45^\circ - \cot^2 45^\circ) - (\cos^2 60^\circ + \sin^2 120^\circ).$$

3. (*a*) Prove that

$$\cos 3A = 4 \cos^3 A - 3 \cos A.$$

(*b*) If $A + B + C = 180^\circ$, prove that

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

4. In a triangle ABC, prove

(*a*) $c = a \cos B + b \cos A$

(*b*) $(b-c) \cos \frac{A}{2} = a \sin \frac{B+C}{2}$.

5. Two vertical poles are 120 feet apart and the height of one is double that of the other. From the middle point of the line joining their feet, an observer finds the angular elevations of their tops to be complementary. Find their heights.

6. Draw the graph of $\cos x$ between the values $x = 0^\circ$ and $x = 360^\circ$ and read off from the graph the value of $\cos 300^\circ$.

1962

1. (*a*) The circular measure of two angles of a triangle are $\frac{1}{2}$ and $\frac{1}{3}$. Find the number of degrees and minutes in the third angle.

[$\frac{22}{7}$ radians $= 2$ right angles].

(*b*) The diameter of a graduated circle is 6 ft. and the graduations on its rim are $15'$ apart; find the distance (in inches correct to two places of decimals) from one graduation to another next to it.

[$\pi = \frac{22}{7}$].

2. (*a*) If A, B, $A+B$ are all acute angles, prove geometrically, that

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

(*b*) Show that

$$\sin 420^\circ \cos 390^\circ + \cos(-300^\circ) \sin(-330^\circ) = 1.$$

3. (*a*) Find the values of θ between 0° and 360° which satisfy the equation

$$\cos^2\theta - \sin\theta = \tfrac{1}{4}.$$

(*b*) If $A+B+C = 180^\circ$, prove that

$$\sin A + \sin B - \sin C = 4 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \cos \tfrac{1}{2}C.$$

4. In a triangle ABC, prove that

(*a*) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$. (*b*) $a \sin\left(\dfrac{A}{2} + C\right) = (b+c)\sin\dfrac{A}{2}$.

5. The angle of elevation of the top of a tower is observed to be 60° from a point in the horizontal plane through the foot of the tower; at a point 40 ft. vertically above the first point of observation, the elevation is found to be 45°. Find the height of the tower and its horizontal distance from the points of observation.

6. Draw the graph of $\cos x$, between the values of $x = -\pi$ and $x = \pi$ and read off from the graph the value of $\cos 120^\circ$.

TABLES OF LOGARITHMS, NATURAL SINES, NATURAL TANGENTS, LOGARITHMIC SINES, LOGARITHMIC TANGENTS ETC

TABLE 1

LOGARITHMS OF NUMBERS

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
10	00000	00432	00860	01284	01703	02119	02531	02938	03342	03743	42	83	125	166	208	219	290	331	373
11	04139	04532	04922	05308	05690	06070	06446	06819	07188	07555	38	76	114	152	190	227	265	302	340
12	07918	08279	08636	08991	09342	09691	10037	10380	10721	11059	35	70	105	140	175	209	243	278	313
13	11394	11727	12057	12385	12710	13033	13354	13672	13988	14301	32	65	97	129	162	193	225	258	290
14	14613	14922	15229	15534	15836	16137	16435	16732	17026	17319	30	60	90	120	150	160	210	240	270
15	17609	17898	18184	18469	18752	19033	19312	19590	19866	20140	28	56	84	112	140	168	196	224	252
16	20412	20683	20952	21219	21484	21748	22011	22272	22531	22789	26	53	79	105	132	158	184	210	237
17	23045	23300	23553	23805	24055	24304	24551	24797	25042	25285	25	50	74	99	124	149	174	199	223
18	25527	25768	26007	26245	26482	26717	26951	27184	27416	27646	23	47	70	94	117	141	164	188	211
19	27875	28103	28330	28556	28780	29003	29226	29447	29667	29885	22	45	67	89	111	134	156	178	201
20	30103	30320	30535	30750	30963	31175	31387	31597	31806	32015	21	42	64	85	106	127	148	170	191
21	32222	32428	32634	32838	33041	33244	33445	33646	33846	34044	20	40	61	81	101	121	141	162	182
22	34242	34439	34635	34830	35025	35218	35411	35603	35793	35984	19	39	58	77	97	116	135	154	174
23	36173	36361	36549	36736	36922	37107	37291	37475	37658	37840	19	37	56	74	93	111	130	148	167
24	38021	38202	38382	38561	38739	38917	39094	39270	39445	39620	18	36	53	71	89	107	124	142	160
25	39794	39967	40140	40312	40483	40654	40824	40993	41162	41330	17	34	51	68	85	102	119	136	153
26	41497	41664	41830	41996	42160	42325	42488	42651	42813	4297[illegible]	16	33	49	66	82	98	115	131	148
27	43136	43297	43457	43616	43775	43933	44091	44248	44404	44560	16	32	47	63	79	95	111	126	142
28	44716	44871	45025	45179	45332	45484	45637	45788	45939	46090	15	30	46	61	76	91	106	122	137
29	46240	46389	46538	46687	46835	46982	47129	47276	47422	47567	15	29	44	59	74	88	103	118	132

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
30	47712	47857	48001	48144	48287	48430	48572	48714	48855	48996	14	29	43	57	72	86	100	114	129
31	49136	49276	49415	49554	49693	49831	49969	50106	50243	50379	14	28	42	55	69	83	97	110	125
32	50515	50651	50786	50920	51055	51188	51322	51455	51587	51720	13	27	40	54	67	80	94	107	121
33	51851	51983	52114	52244	52375	52504	52634	52763	52892	53020	13	26	39	52	65	78	91	104	117
34	53148	53275	53403	53529	53656	53782	53908	54033	54158	54283	13	25	38	50	63	76	88	101	113
35	54407	54531	54654	54777	54900	55023	55145	55267	55388	55509	12	24	37	49	61	73	86	98	110
36	55630	55751	55871	55991	56110	56229	56348	56467	56585	56703	12	24	36	48	60	71	83	95	107
37	56820	56937	57054	57171	57287	57403	57519	57634	57749	57864	12	23	35	46	58	70	81	93	104
38	57978	58092	58206	58320	58433	58546	58659	58771	58883	58995	11	23	34	45	57	68	79	90	102
39	59106	59218	59329	59439	59550	59660	59770	59879	59988	60097	11	22	33	44	55	66	77	88	99
40	60206	60314	60423	60531	60638	60746	60853	60959	61066	61172	11	21	32	43	54	64	75	86	97
41	61278	61384	61490	61595	61700	61805	61909	62014	62118	62221	10	21	31	42	52	63	73	84	94
42	62325	62428	62531	62634	62737	62839	62941	63043	63144	63246	10	20	31	41	51	61	71	82	92
43	63347	63448	63548	63649	63749	63849	63949	64048	64147	64246	10	20	30	40	50	60	70	80	90
44	64345	64444	64542	64640	64738	64836	64933	65031	65128	65225	10	20	29	39	49	59	68	78	88
45	65321	65418	65514	65610	65706	65801	65896	65992	66087	66181	10	19	29	38	48	57	67	76	86
46	66276	66370	66464	66558	66652	66745	66839	66932	67025	67117	9	19	28	37	47	56	65	75	84
47	67210	67302	67394	67486	67578	67669	67761	67852	67943	68034	9	18	27	37	46	55	64	73	82
48	68124	68215	68305	68395	68485	68574	68664	68753	68842	68931	9	18	27	36	45	53	62	71	80
49	69020	69108	69197	69285	69373	69461	69548	69636	69723	69810	9	18	26	35	44	53	61	70	79
50	69897	69984	70070	70157	70243	70329	70415	70501	70586	70672	9	17	26	34	43	52	60	69	77
51	70757	70842	70927	71012	71096	71181	71265	71349	71433	71517	8	17	25	34	42	51	59	67	76
52	71600	71684	71767	71850	71933	72016	72099	72181	72263	72346	8	17	25	33	42	50	58	66	75
53	72428	72509	72591	72673	72754	72835	72916	72997	73078	73159	8	16	24	32	41	49	57	65	73
54	73239	73320	73400	73480	73560	73640	73719	73799	73878	73957	8	16	24	32	40	48	56	64	72

LOGARITHMS OF NUMBERS

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
55	74036	74115	74194	74273	74351	74429	74507	74586	74663	74741	8	16	23	31	39	47	55	62	70
56	74819	74896	74974	75051	75128	75205	75282	75358	75435	75511	8	15	23	31	39	46	54	62	69
57	75587	75664	75740	75815	75891	75967	76042	76118	76193	76268	8	15	23	30	38	45	53	60	68
58	76343	76418	76492	76567	76641	76716	76790	76864	76938	77012	7	15	22	30	37	45	52	59	67
59	77085	77159	77232	77305	77379	77452	77525	77597	77670	77743	7	15	22	29	37	44	51	58	66
60	77815	77887	77960	78032	78104	78176	78247	78319	78390	78462	7	14	22	29	36	43	50	57	65
61	78533	78604	78675	78746	78817	78888	78958	79029	79099	79169	7	14	21	28	35	42	49	56	64
62	79239	79309	79379	79449	79518	79588	79657	79727	79796	79865	7	14	21	28	35	42	49	56	63
63	79934	80003	80072	80140	80209	80277	80346	80414	80482	80550	7	14	21	27	34	41	48	55	62
64	80618	80686	80754	80821	80889	80956	81023	81090	81158	81224	7	13	20	27	34	40	47	54	60
65	81291	81358	81425	81491	81558	81624	81690	81757	81823	81889	7	13	20	26	33	40	46	53	59
66	81954	82020	82086	82151	82217	82282	82347	82413	82478	82543	7	13	20	26	33	39	46	52	59
67	82607	82672	82737	82802	82866	82930	82995	83059	83123	83187	6	13	19	26	32	39	45	52	58
68	83251	83315	83378	83442	83506	83569	83632	83696	83759	83822	6	13	19	25	32	38	44	50	57
69	83885	83948	84011	84073	84136	84198	84261	84323	84386	84448	6	13	19	25	31	37	44	50	56
70	84510	84572	84634	84696	84757	84819	84880	84942	85003	85065	6	12	18	25	31	37	43	49	55
71	85126	85187	85248	85309	85370	85431	85491	85552	85612	85673	6	12	18	24	30	36	42	49	55
72	85733	85794	85854	85914	85974	86034	86094	86153	86213	86273	6	12	18	24	30	36	42	48	54
73	86332	86392	86451	86510	86570	86629	86688	86747	86806	86864	6	12	18	24	30	35	41	47	53
74	86923	86982	87040	87099	87157	87216	87274	87332	87390	87448	6	12	18	23	29	35	41	47	52

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
75	87506	87564	87622	87679	87737	87795	87852	87910	87967	88024	6	12	17	23	29	35	40	46	52
76	88081	88138	88195	88252	88309	88366	88423	88480	88536	88593	6	11	17	23	29	34	40	46	51
77	88649	88705	88762	88818	88874	88930	88986	89042	89098	89154	6	11	17	22	28	34	39	45	50
78	89209	89265	89321	89376	89432	89487	89542	89597	89653	89708	6	11	17	22	28	33	39	44	50
79	89763	89818	89873	89927	89982	90037	90091	90146	90200	90255	5	11	16	22	27	33	38	44	49
80	90309	90363	90417	90472	90526	90580	90634	90687	90741	90795	5	11	16	22	27	32	38	43	49
81	90849	90902	90956	91009	91062	91116	91169	91222	91275	91328	5	11	16	21	27	32	37	43	48
82	91381	91434	91487	91540	91593	91645	91698	91751	91803	91855	5	11	16	21	26	32	37	42	47
83	91908	91960	92012	92065	92117	92169	92221	92273	92324	92376	5	10	16	21	26	31	36	42	47
84	92428	92480	92531	92583	92634	92686	92737	92788	92840	92891	5	10	15	21	26	31	36	41	46
85	92942	92993	93044	93095	93146	93197	93247	93298	93349	93399	5	10	15	20	25	30	36	41	46
86	93450	93500	93551	93601	93651	93702	93752	93802	93852	93902	5	10	15	20	25	30	35	40	45
87	93952	94002	94052	94101	94151	94201	94250	94300	94349	94399	5	10	15	20	25	30	35	40	45
88	94448	94498	94547	94596	94645	94694	94743	94792	94841	94890	5	10	15	20	25	29	34	39	44
89	94939	94988	95036	95085	95134	95182	95231	95279	95328	95376	5	10	15	19	24	29	34	39	44
90	95424	95472	95521	95569	95617	95665	95713	95761	95809	95856	5	10	14	19	24	29	34	38	43
91	95904	95952	95999	96047	96095	96142	96190	96237	96284	96332	5	9	14	19	24	29	33	38	43
92	96379	96426	96473	96520	96567	96614	96661	96708	96755	96802	5	9	14	19	24	28	33	38	42
93	96848	96895	96942	96988	97035	97081	97128	97174	97220	97267	5	9	14	19	23	28	33	37	42
94	97313	97359	97405	97451	97497	97543	97589	97635	97681	97727	5	9	14	18	23	28	32	37	41
95	97772	97818	97864	97909	97955	98000	98046	98091	98137	98182	5	9	14	18	23	27	32	36	41
96	98227	98272	98318	98363	98408	98453	98498	98543	98588	98632	5	9	14	18	23	27	32	36	41
97	98677	98722	98767	98811	98856	98900	98945	98989	99034	99078	4	9	13	18	22	27	31	36	40
98	99123	99167	99211	99255	99300	99344	99388	99432	99476	99520	4	9	13	18	22	26	31	35	40
99	99564	99607	99651	99695	99739	99782	99826	99870	99913	99957	4	9	13	17	22	26	30	35	39

TABLE II
NATURAL SINES

	0′	10′	20′	30′	40′	50′	60′		Mean Differences 1′	2′	3′	4′	5′	6′	7′	8′	9′
0°	0·00000	0·00291	0·00582	0·00873	0·01164	0·01454	0·01745	89°	29	58	87	116	145	175	204	233	262
1°	·01745	·02036	·02327	·02618	·02908	·03199	·03490	88°	29	58	87	116	145	175	204	233	262
2°	·03490	·03781	·04071	·04362	·04653	·04943	·05234	87°	29	58	87	116	145	175	204	233	262
3°	·05234	·05524	·05814	·06105	·06395	·06685	·06976	86°	29	58	87	116	145	174	203	232	261
4°	·06976	·07266	·07556	·07846	·08136	·08426	·08716	85°	29	58	87	116	145	174	203	232	261
5°	0·08716	0·09005	0·09295	0·09585	0·09874	0·10164	0·10453	84°	29	58	87	116	145	174	203	232	261
6°	·10453	·10742	·11031	·11320	·11609	·11898	·12187	83°	29	58	87	116	145	174	203	232	261
7°	·12187	·12476	·12764	·13053	·13341	·13629	·13917	82°	29	58	87	116	145	173	202	231	260
8°	·13917	·14205	·14493	·14781	·15069	·15356	·15643	81°	29	58	86	115	144	173	202	230	259
9°	·15643	·15931	·16218	·16505	·16792	·17078	·17365	80°	29	57	86	115	144	172	201	230	258
10°	0·17365	0·17651	0·17937	0·18224	0·18509	0·18795	0·19081	79°	29	57	86	115	144	172	201	229	258
11°	·19081	·19366	·19652	·19937	·20222	·20507	·20791	78°	29	57	86	114	143	171	200	228	257
12°	·20791	·21076	·21360	·21644	·21928	·22212	·22495	77°	28	57	85	114	142	170	199	227	256
13°	·22495	·22778	·23062	·23345	·23627	·23910	·24192	76°	28	57	85	113	141	170	198	226	255
14°	·24192	·24474	·24756	·25038	·25320	·25601	·25882	75°	28	56	85	113	141	169	197	226	254
15°	0·25882	0·26163	0·26443	0·26724	0·27004	0·27284	0·27564	74°	28	56	84	112	140	168	196	224	252
16°	·27564	·27843	·28123	·28402	·28680	·28959	·29237	73°	28	56	84	112	140	167	195	223	251
17°	·29237	·29515	·29793	·30071	·30348	·30625	·30902	72°	28	56	83	111	139	166	194	222	250
18°	·30902	·31178	·31454	·31730	·32006	·32282	·32557	71°	28	55	83	110	138	166	193	221	248
19°	·32557	·32832	·33106	·33381	·33655	·33929	·34202	70°	27	55	82	110	137	164	192	219	247

20°	0·34202	0·34475	0·34748	0·35021	0·35293	0·35565	0·35837	69°	27	55	82	109	137	164	191	218	246
21°	·35837	·36108	·36379	·36650	·36921	·37191	·37461	68°	27	54	81	108	136	163	190	217	244
22°	·37461	·37730	·37999	·38268	·38537	·38805	·39073	67°	27	54	81	108	135	161	188	215	242
23°	·39073	·39341	·39608	·39875	·40142	·40408	·40674	66°	27	53	80	107	134	160	187	214	240
24°	·40674	·40939	·41204	·41469	·41734	·41998	·42262	65°	27	53	80	106	133	159	186	212	238
25°	0·42262	0·42525	0·42788	0·43051	0·43313	0·43575	0·43837	64°	26	52	79	105	131	157	184	210	236
26°	·43837	·44098	·44359	·44620	·44880	·45140	·45399	63°	26	52	78	104	130	156	182	208	234
27°	·45399	·45658	·45917	·46175	·46433	·46690	·46947	62°	26	52	77	103	129	155	181	206	232
28°	·46947	·47204	·47460	·47716	·47971	·48226	·48481	61°	26	51	77	102	128	154	179	204	230
29°	·48481	·48735	·48989	·49242	·49495	·49748	·50000	60°	25	51	76	101	127	152	177	202	228
30°	0·50000	0·50252	0·50503	0·50754	0·51004	0·51254	0·51504	59°	25	50	75	100	125	150	175	200	225
31°	·51504	·51753	·52002	·52250	·52498	·52745	·52992	58°	25	50	74	99	124	149	174	198	223
32°	·52992	·53238	·53484	·53730	·53975	·54220	·54464	57°	25	49	74	98	123	147	172	196	221
33°	·54464	·54708	·54951	·55194	·55436	·55678	·55919	56°	24	49	73	97	122	146	170	194	219
34°	·55919	·56160	·56401	·56641	·56880	·57119	·57358	55°	24	48	72	96	120	144	168	192	216
35°	0·57358	0·57596	0·57833	0·58070	0·58307	0 58543	0·58779	54°	24	47	71	95	119	142	166	190	213
36°	·58779	·59014	·59248	·59482	·59716	·59949	·60182	53°	23	47	70	94	117	140	164	187	211
37°	·60182	·60414	·60645	·60876	·61107	·61337	·61566	52°	23	46	70	92	116	139	162	185	208
38°	·61566	·61795	·62024	·62251	·62479	·62706	·62932	51°	23	46	68	91	114	137	159	182	205
39°	·62932	·63158	·63383	·63608	·63832	·64056	·64279	50°	22	45	67	90	112	135	157	179	202
40°	0·64279	0·64501	0·64723	0·64945	0·65166	0·65386	0·65606	49°	22	44	66	88	111	133	155	177	199
41°	·65606	·65825	·66044	·66262	·66480	·66697	·66913	48°	22	44	65	87	109	131	153	174	196
42°	·66913	·67129	·67344	·67559	·67773	·67987	·68200	47°	21	43	64	86	107	129	150	172	193
43°	·68200	·68412	·68624	·68835	·69046	·69256	·69466	46°	21	42	63	84	106	127	148	169	190
44°	·69466	·69675	·69883	·70091	·70298	·70505	·70711	45°	21	42	62	83	104	124	145	166	187
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

NATURAL COSINES

NATURAL SINES

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences								
45°	0·70711	0·70916	0·71121	0·71325	0·71529	0·71732	0·71934	44°	20	41	61	82	102	122	143	163	184
46°	·71934	·72136	·72337	·72537	·72737	·72937	·73135	43°	20	40	60	80	100	120	140	160	180
47°	·73135	·73333	·73531	·73728	·73924	·74120	·74314	42°	20	39	59	78	98	118	138	157	177
48°	·74314	·74509	·74703	·74896	·75088	·75280	·75471	41°	19	39	58	77	96	116	135	154	173
49°	·75471	·75661	·75851	·76041	·76229	·76417	·76604	40°	19	38	57	76	95	113	132	151	170
50°	0·76604	0·76791	0·76977	0·77162	0·77347	0·77531	0·77715	39°	19	37	56	74	93	111	130	148	167
51°	·77715	·77897	·78079	·78261	·78442	·78622	·78801	38°	18	36	54	72	91	109	127	145	163
52°	·78801	·78980	·79158	·79335	·79512	·79688	·79864	37°	18	35	53	71	89	106	124	142	159
53°	·79864	·80038	·80212	·80386	·80558	·80730	·80902	36°	17	35	52	69	87	104	121	138	156
54°	·80902	·81072	·81242	·81412	·81580	·81748	·81915	35°	17	34	51	68	85	101	118	135	152
55°	0·81915	0·82082	0·82248	0·82413	0·82577	0·82741	0·82904	34°	16	33	49	66	82	99	115	132	148
56°	·82904	·83066	·83228	·83389	·83549	·83708	·83867	33°	16	32	48	64	80	96	112	128	144
57°	·83867	·84025	·84182	·84339	·84495	·84650	·84805	32°	16	31	47	63	78	94	110	125	141
58°	·84805	·84959	·85112	·85264	·85416	·85567	·85717	31°	15	30	46	61	76	91	106	122	137
59°	·85717	·85866	·86015	·86163	·86310	·86457	·86603	30°	15	30	44	59	74	89	103	118	133
60°	0·86603	0·86748	0·86892	0·87036	0·87178	0·87321	0·87462	29°	14	29	43	57	72	86	100	114	129
61°	·87462	·87603	·87743	·87882	·88020	·88158	·88295	28°	14	28	42	55	69	83	97	111	125
62°	·88295	·88431	·88566	·88701	·88835	·88968	·89101	27°	13	27	40	54	67	81	94	108	121
63°	·89101	·89232	·89363	·89493	·89623	·89752	·89879	26°	13	26	39	52	65	78	91	104	117
64°	·89879	·90007	·90133	·90259	·90383	·90507	·90631	25°	13	25	38	50	63	75	88	100	113

	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′
65°	0·90631	0·90753	0·90875	0·90996	0·91116	0·91236	0·91355	24°	12	24	36	48	60	72	84	96	108
66°	·91355	·91472	·91590	·91706	·91822	·91936	·92050	23°	12	23	35	46	58	70	81	93	104
67°	·92050	·92164	·92276	·92388	·92499	·92609	·92718	22°	11	22	33	45	56	67	78	89	100
68°	·92718	·92827	·92935	·93042	·93148	·93253	·93358	21°	11	21	32	43	53	64	75	85	96
69°	·93358	·93462	·93565	·93667	·93769	·93869	·93969	20°	10	20	31	41	51	61	71	81	92
70°	0·93969	0·94068	0·94167	0·94264	0·94361	0·94457	0·94552	19°	10	19	29	39	49	58	68	78	87
71°	·94552	·94646	·94740	·94832	·94924	·95015	·95106	18°	9	18	28	37	46	55	64	74	83
72°	·95106	·95195	·95284	·95372	·95459	·95545	·95630	17°	9	18	26	35	44	52	61	70	79
73°	·95630	·95715	·95799	·95882	·95964	·96046	·96126	16°	8	17	25	33	41	50	58	66	74
74°	·96126	·96206	·96285	·96363	·96440	·96517	·96593	15°	8	16	23	31	39	47	54	62	70
75°	0·96593	0·96667	0·96742	0·96815	0·96887	0·96959	0·97030	14°	7	15	22	29	36	44	51	58	65
76°	·97030	·97100	·97169	·97237	·97304	·97371	·97437	13°	7	14	20	27	34	41	47	54	61
77°	·97437	·97502	·97566	·97630	·97692	·97754	·97815	12°	6	13	19	25	32	38	44	50	57
78°	·97815	·97875	·97934	·97992	·98050	·98107	·98163	11°	6	12	17	23	29	35	41	46	52
79°	·98163	·98218	·98272	·98325	·98378	·98430	·98481	10°	5	11	16	21	27	32	37	42	48
80°	0·98481	0·98531	0·98580	0·98629	0·98676	0·98723	0·98769	9°	5	10	14	19	24	29	34	38	43
81°	·98769	·98814	·98858	·98902	·98944	·98986	·99027	8°	4	9	13	17	22	26	30	34	39
82°	·99027	·99067	·99106	·99144	·99182	·99219	·99255	7°	4	8	11	15	19	23	27	30	34
83°	·99255	·99290	·99324	·99357	·99390	·99421	·99452	6°	3	7	10	13	17	20	23	26	30
84°	·99452	·99482	·99511	·99540	·99567	·99594	·99619	5°	3	6	8	11	14	17	20	22	25
85°	0·99619	0·99644	0·99668	0·99692	0·99714	0·99736	0·99756	4°	2	5	7	9	12	14	16	18	21
86°	·99756	·99776	·99795	·99813	·99831	·99847	·99863	3°	2	4	5	7	9	11	13	14	16
87°	·99863	·99878	·99892	·99905	·99917	·99929	·99939	2°	1	3	4	5	7	8	9	10	12
88°	·99939	·99949	·99958	·99966	·99973	·99979	·99985	1°									
89°	·99985	·99989	·99993	·99996	·99998	1·00000	1·00000	0°									
90°	1·00000																

NATURAL COSINES

TABLE III

NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences								
0°	0·00000	0·00291	0·00582	0·00873	0·01164	0·01455	0·01746	89°	29	58	87	116	146	175	204	233	262
1°	·01746	·02037	·02328	·02619	·02910	·03201	·03492	88°	29	58	87	116	146	175	204	233	262
2°	·03492	·03783	·04075	·04366	·04658	·04949	·05241	87°	29	58	87	116	146	175	204	233	262
3°	·05241	·05533	·05324	·06116	·06408	·06700	·06993	86°	29	58	88	117	146	175	204	234	263
4°	·06993	·07285	·07578	·07870	·08163	·08456	·08749	85°	29	58	88	117	146	175	204	234	263
5°	0·08749	0·09042	0·09335	0·09629	0·09923	0·10216	0·10510	84°	29	59	88	118	147	176	206	235	265
6°	·10510	·10805	·11099	·11394	·11688	·11983	·12278	83°	29	59	88	118	147	176	206	235	265
7°	·12278	·12574	·12869	·13165	·13461	·13758	·14054	82°	30	59	89	118	148	178	207	237	266
8°	·14054	·14351	·14648	·14945	·15243	·15540	·15838	81°	30	59	89	119	149	178	208	238	267
9°	·15838	·16137	·16435	·16734	·17033	·17333	·17633	80°	30	60	90	120	150	179	209	239	269
10°	0·17633	0·17933	0·18233	0·18534	0·18835	0·19136	0·19438	79°	30	60	90	120	151	181	211	241	271
11°	·19438	·19740	·20042	·20345	·20648	·20952	·21256	78°	30	61	91	121	152	182	212	242	273
12°	·21256	·21560	·21864	·22169	·22475	·22781	·23087	77°	31	61	92	122	153	183	214	244	275
13°	·23087	·23393	·23700	·24008	·24316	·24624	·24933	76°	31	62	92	123	154	185	216	246	277
14°	·24933	·25242	·25552	·25862	·26172	·26483	·26795	75°	31	62	93	124	155	186	217	248	279
15°	0·26795	0·27107	0·27419	0·27732	0·28046	0·28360	0·28675	74°	31	63	94	125	157	188	219	250	282
16°	·28675	·28990	·29305	·29621	·29938	·30255	·30573	73°	32	63	95	126	158	190	221	253	285
17°	·30573	·30891	·31210	·31530	·31850	·32171	·32492	72°	32	64	96	128	160	192	224	256	288
18°	·32492	·32814	·33136	·33460	·33783	·34108	·34433	71°	32	65	97	129	162	194	226	259	291
19°	·34433	·34758	·35085	·35412	·35740	·36068	·36397	70°	33	65	98	131	164	196	229	262	294

20°	0·36397	0·36727	0·37057	0·37388	0·37720	0·38053	0·38386	69°	33	66	100	133	166	199	232	265	298
21°	·38386	·38721	·39055	·39391	·39727	·40065	·40403	68°	34	67	101	134	168	202	236	269	302
22°	·40403	·40741	·41081	·41421	·41763	·42105	·42447	67°	34	68	102	136	170	205	239	273	306
23°	·42447	·42791	·43136	·43481	·43828	·44175	·44523	66°	35	69	104	138	173	208	242	277	311
24°	·44523	·44872	·45222	·45573	·45924	·46277	·46631	65°	35	70	105	140	176	211	246	281	316
25°	0·46631	0·46985	0·47341	0·47698	0·48055	0·48414	0·48773	64°	36	71	107	143	179	214	250	286	321
26°	·48773	·49134	·49495	·49858	·50222	·50587	·50953	63°	36	73	109	145	182	218	254	291	327
27°	·50953	·51320	·51688	·52057	·52427	·52798	·53171	62°	37	74	111	148	185	222	259	296	333
28°	·53171	·53545	·53920	·54296	·54673	·55051	·55431	61°	38	75	113	151	189	226	264	302	339
29°	·55431	·55812	·56194	·56577	·56962	·57348	·57735	60°	38	77	115	154	192	230	269	307	346
30°	0·57735	0·58124	0·58513	0·58905	0·59297	0·59691	0·60086	59°	39	78	118	157	196	235	274	313	353
31°	·60086	·60483	·60881	·61280	·61681	·62083	·62487	58°	40	80	120	160	200	240	280	320	360
32°	·62487	·62892	·63299	·63707	·64117	·64528	·64941	57°	41	82	123	164	205	245	286	327	368
33°	·64941	·65355	·65771	·66189	·66608	·67028	·67451	56°	42	84	126	167	209	251	293	334	376
34°	·67451	·67875	·68301	·68728	·69157	·69588	·70021	55°	43	86	128	171	214	257	300	342	385
35°	0·70021	0·70455	0·70891	0·71329	0·71769	0·72211	0·72654	54°	44	88	132	176	220	263	307	351	395
36°	·72654	·73100	·73547	·73996	·74447	·74900	·75355	53°	45	90	135	180	225	270	315	360	405
37°	·75355	·75812	·76272	·76733	·77196	·77661	·78129	52°	46	92	139	185	231	277	324	370	416
38°	·78129	·78598	·79070	·79544	·80020	·80498	·80978	51°	48	95	143	190	238	285	333	380	428
39°	·80978	·81461	·81946	·82434	·82923	·83415	·83910	50°	49	98	147	196	245	293	342	391	440
40°	0·83910	0·84407	0·84906	0·85408	0·85912	0·86419	0·86929	49°	50	101	151	201	252	302	352	402	453
41°	·86929	·87441	·87955	·88473	·88992	·89515	·90040	48°	52	104	156	208	260	311	363	415	467
42°	·90040	·90569	·91099	·91633	·92170	·92709	·93252	47°	54	107	161	214	268	321	375	429	482
43°	·93252	·93797	·94345	·94896	·95451	·96008	·96569	46°	55	111	166	221	277	332	387	442	498
44°	·96569	·97133	·97700	·98270	·98843	·99420	1·00000	45°	57	114	172	229	286	343	400	457	515
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

NATURAL COTANGENTS

NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences								
45°	1·00000	1·00583	1·01170	1·01761	1·02355	1·02952	1·03553	44°	59	118	178	237	296	355	414	474	533
46°	·03553	·04158	·04766	·05378	·05994	·06613	·07237	43°	61	123	184	246	307	368	430	491	553
47°	·07237	·07864	·08496	·09131	·09770	·10414	·11061	42°	64	127	191	255	319	382	446	510	573
48°	·11061	·11713	·12369	·13029	·13694	·14363	·15037	41°	66	132	199	265	332	397	463	530	596
49°	·15037	·15715	·16398	·17085	·17777	·18474	·19175	40°	69	138	207	276	345	413	482	552	620
50°	1·19175	1·19882	1·20593	1·21310	1·22031	1·22758	1·23490	39°	72	144	216	288	360	431	503	575	647
51°	·23490	·24227	·24969	·25717	·26471	·27[illegible]	·27994	38°	75	150	225	300	376	451	526	601	676
52°	·27994	·28764	·29541	·30323	·31110	·31904	·32704	37°	78	157	235	314	392	471	549	628	707
53°	·32704	·33511	·34323	·35142	·35968	·36800	·37638	36°	82	164	247	329	411	493	576	658	740
54°	·37638	·38484	·39336	·40195	·41061	·41934	·42815	35°	86	172	259	345	431	517	603	690	776
55°	1·42815	1·43703	1·44598	1·45501	1·46411	1·47330	1·48256	34°	91	181	272	363	453	544	634	725	816
56°	·48256	·49190	·50133	·51084	·52043	·53010	·53987	33°	96	191	287	382	478	573	669	764	860
57°	·53987	·54972	·55966	·56969	·57981	·59002	·60033	32°	101	201	302	403	504	604	705	806	907
58°	·60033	·61074	·62125	·63185	·64256	·65337	·66428	31°	107	213	320	426	533	639	746	852	959
59°	·66428	·67530	·68643	·69766	·70901	·72047	·73205	30°	113	226	339	451	565	677	790	903	1016
60°	1·7321	1·7437	1·7556	1·7675	1·7796	1·7917	1·8040	29°	12	24	36	48	60	72	84	96	108
61°	1·8040	1·8165	1·8291	1·8418	1·8546	1·8676	1·8807	28°	13	25	38	51	64	77	89	102	115
62°	1·8807	1·8940	1·9074	1·9210	1·9347	1·9486	1·9626	27°	14	27	41	54	68	82	95	109	122
63°	1·9626	1·9768	1·9912	2·0057	2·0204	2·0353	2·0503	26°	15	29	44	58	73	88	102	117	131
64°	2·0503	2·0655	2·0809	2·0965	2·1123	2·1283	2·1445	25°	16	31	47	63	79	94	110	126	141

	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′
65°	2·1445	2·1609	2·1775	2·1943	2·2113	2·2286	2·2460	24°	17	34	51	68	85	101	118	135	152
66°	2·2460	2·2637	2·2817	2·2998	2·3183	2·3369	2·3559	23°	18	37	55	73	92	110	128	146	165
67°	2·3559	2·3750	2·3945	2·4142	2·4342	2·4545	2·4751	22°	20	40	60	80	100	119	139	159	179
68°	2·4751	2·4960	2·5172	2·5386	2·5605	2·5826	2·6051	21°	22	43	65	87	109	130	152	174	195
69°	2·6051	2·6279	2·6511	2·6746	2·6985	2·7228	2·7475	20°	24	47	71	95	119	142	166	190	213
70°	2·7475	2·7725	2·7980	2·8239	2·8502	2·8770	2·9042	19°	26	52	78	104	131	157	183	209	235
71°	2·9042	2·9319	2·9600	2·9887	3·0178	3·0475	3·0777	18°	29	58	87	116	145	174	202	231	260
72°	3·0777	3·1084	3·1397	3·1716	3·2041	3·2371	3·2709	17°	32	64	97	129	161	193	225	258	290
73°	3·2709	3·3052	3·3402	3·3759	3·4124	3·4495	3·4874	16°	36	72	108	144	181	216	253	289	325
74°	3·4874	3·5261	3·5656	3·6059	3·6470	3·6891	3·7321	15°	41	81	122	163	204	244	285	326	366
75°	3·7321	3·7760	3·8208	3·8667	3·9136	3·9617	4·0108	14°	46	93	139	185	232	278	325	371	418
76°	4·0108	4·0611	4·1126	4·1653	4·2193	4·2747	4·3315	13°	53	107	160	214	267	320	374	427	481
77°	4·3315	4·3897	4·4494	4·5107	4·5736	4·6382	4·7046	12°	62	124	186	248	311	373	435	497	559
78°	4·7046	4·7729	4·8430	4·9152	4·9894	5·0658	5·1446	11°	73	146	220	293	366	439	512	586	659
79°	5·1446	5·2257	5·3093	5·3955	5·4845	5·5764	5·6713	10°	88	175	263	350	438	526	613	701	788
80°	5·6713	5·7694	5·8708	5·9758	6·0844	6·1970	6·3138	9°	The differences change very rapidly here so that they cannot be tabulated.								
81°	6·3138	6·4348	6·5606	6·6912	6·8269	6·9682	7·1154	8°									
82°	7·1154	7·2687	7·4287	7·5958	7·7704	7·9530	8·1443	7°									
83°	8·1443	8·3450	8·5555	8·7769	9·0098	9·2553	9·5144	6°									
84°	9·5144	9·7882	10·0780	10·3854	10·7119	11·0594	11·4301	5°									
85°	11·4301	11·8262	12·2505	12·7062	13·1969	13·7267	14·3007	4°	The cotangent of a small angle of x' or the tangent of $90°-x'$ is very nearly equal to 3437·7 divided by x.								
86°	14·3007	14·9244	15·6048	16·3499	17·1693	18·0750	19·0811	3°									
87°	19·0811	20·2056	21·4704	22·9038	24·5418	26·4316	28·6363	2°									
88°	28·6363	31·2416	34·3678	38·1885	42·9641	49·1039	57·2900	1°									
89°	57·2900	68·7501	85·9398	114·589	171·885	343·774	$+\infty$	0°									
90°	$+\infty$																
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

NATURAL COTANGENTS

TABLE IV

LOGARITHMIC SINES

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
0°	$-\infty$	7·46373	7·76475	7·94084	8·06578	8·16268	8·24186	89°									
1°	8·24186	8·30879	8·36678	8·41792	8·46366	8·50504	8·54282	88°									
2°	8·54282	8·57757	8·60973	8·63968	8·66769	8·69400	8·71880	87°									
3°	8·71880	8·74226	8·76451	8·78568	8·80585	8·82513	8·84358	86°									
4°	8·84358	8·86128	8·87829	8·89464	8·91040	8·92561	8·94030	85°									
5°	8·94030	8·95450	8·96825	8·98157	8·99450	9·00704	9·01923	84°									
6°	9·01923	9·03109	9·04262	9·05386	9·06481	9·07548	9·08589	83°									
7°	9·08589	9·09606	9·10599	9·11570	9·12519	9·13447	9·14356	82°	96	192	288	384	480	576	672	768	864
8°	9·14356	9·15245	9·16116	9·16970	9·17807	9·18628	9·19433	81°	85	169	254	338	423	507	592	676	761
9°	9·19433	9·20223	9·20999	9·21761	9·22509	9·23244	9·23967	80°	76	151	227	302	378	453	529	604	680
10°	9·23967	9·24677	9·25376	9·26063	9·26739	9·27405	9·28060	79°	68	136	204	272	341	409	477	545	613
11°	·28060	·28705	·29340	·29966	·30582	·31189	·31788	78°	62	124	186	248	310	373	435	497	559
12°	·31788	·32378	·32960	·33534	·34100	·34658	·35209	77°	57	114	171	228	285	342	399	456	513
13°	·35209	·35752	·36289	·36819	·37341	·37858	·38368	76°	53	105	158	210	263	316	368	421	473
14°	·38368	·38871	·39369	·39860	·40346	·40825	·41300	75°	49	98	147	195	244	293	342	391	440
15°	9·41300	9·41768	9·42232	9·42690	9·43143	9·43591	9·44034	74°	46	91	137	182	228	273	319	364	410
16°	·44034	·44472	·44905	·45334	·45758	·46178	·46594	73°	43	85	128	171	213	256	299	341	384
17°	·46594	·47005	·47411	·47814	·48213	·48607	·48998	72°	40	80	120	160	201	241	281	321	361
18°	·48998	·49385	·49768	·50148	·50523	·50896	·51264	71°	38	76	113	151	189	227	264	302	340
19°	·51264	·51269	·51991	·52350	·52705	·53056	·53405	70°	36	71	107	143	179	214	250	285	321

Mean Differences (1′–9′): Differences vary so rapidly here that tabulation is impossible. For small angles of x minutes $\log \sin x'$ or $\log \cos (90^\circ - x') = \log x + \bar{4}{\cdot}46373$.

	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′
20°	9·53405	9·53751	9·54093	9·54433	9·54769	9·55102	9·55433	69°	34	68	101	135	169	203	237	270	304
21°	·55433	·55761	·56085	·56408	·56727	·57044	·57358	68°	32	64	96	128	161	193	225	257	289
22°	·57358	·57669	·57976	·58284	·58588	·58889	·59188	67°	31	61	92	122	153	183	214	244	275
23°	·59188	·59484	·59778	·60070	·60359	·60646	·60931	66°	29	58	87	116	146	174	204	233	262
24°	·60931	·61214	·61494	·61773	·62049	·62323	·62595	65°	28	56	83	111	139	166	195	222	250
25°	9·62595	9·62865	9·63133	9·63398	9·63662	9·63924	9·64184	64°	27	53	80	106	133	159	186	212	239
26°	·64184	·64442	·64698	·64953	·65205	·65456	·65705	63°	25	51	76	102	127	152	178	203	229
27°	·65705	·65952	·66197	·66441	·66682	·66922	·67161	62°	24	49	73	97	122	146	170	194	219
28°	·67161	·67398	·67633	·67866	·68098	·68328	·68557	61°	23	47	70	93	117	140	163	186	210
29°	·68557	·68784	·69010	·69234	·69456	·69677	·69897	60°	22	45	67	89	112	134	156	179	201
30°	9·69897	9·70115	9·70332	9·70547	9·70761	9·70973	9·71184	59°	22	43	65	86	107	129	150	172	193
31°	·71184	·71393	·71602	·71809	·72014	·72218	·72421	58°	21	41	62	82	103	124	144	165	185
32°	·72421	·72622	·72823	·73022	·73219	·73416	·73611	57°	20	40	59	79	99	119	139	159	178
33°	·73611	·73805	·73997	·74189	·74379	·74568	·74756	56°	19	38	57	76	96	115	134	153	172
34°	·74756	·74943	·75128	·75313	·75496	·75678	·75859	55°	18	37	55	74	92	110	129	147	165
35°	9·75859	9·76039	9·76218	9·76395	9·76572	9·76747	9·76922	54°	18	35	53	71	89	106	124	142	159
36°	·76922	·77095	·77268	·77439	·77609	·77778	·77946	53°	17	34	51	68	86	103	120	137	154
37°	·77946	·78113	·78280	·78445	·78609	·78772	·78934	52°	17	33	50	66	83	99	116	132	149
38°	·78934	·79095	·79256	·79415	·79573	·79731	·79887	51°	16	32	48	64	80	95	112	127	143
39°	·79887	·80043	·80197	·80351	·80504	·80656	·80807	50°	15	31	46	62	77	92	108	123	138
40°	9·80807	9·80957	9·81106	9·81254	9·81402	9·81549	9·81694	49°	15	30	44	59	74	89	104	118	133
41°	·81694	·81839	·81983	·82126	·82269	·82410	·82551	48°	14	29	43	57	72	86	100	114	129
42°	·82551	·82691	·82830	·82968	·83106	·83242	·83378	47°	14	28	41	55	69	83	97	110	124
43°	·83378	·83513	·83648	·83781	·83914	·84046	·84177	46°	13	27	40	53	67	80	93	106	120
44°	·84177	·84308	·84437	·84566	·84694	·84822	·84949	45°	13	26	38	51	64	77	90	102	115

LOGARITHMIC COSINES

LOGARITHMIC SINES

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences								
45°	9·84949	9·85074	9·85200	9·85324	9·85448	9·85571	9·85693	44°	12	25	37	50	62	74	87	99	112
46°	·85693	·85815	·85936	·86056	·86176	·86294	·86413	43°	12	24	36	48	60	72	84	96	108
47°	·86413	·86530	·86647	·86763	·86879	·86993	·87107	42°	12	23	35	46	58	70	81	93	104
48°	·87107	·87221	·87334	·87446	·87557	·87668	·87778	41°	11	22	34	45	56	67	78	89	100
49°	·87778	·87887	·87996	·88105	·88212	·88319	·88425	40°	11	22	32	43	54	65	76	86	97
50°	9·88425	9·88531	9·88636	9·88741	9·88844	9·88948	9·89050	39°	10	21	31	42	52	62	73	83	94
51°	·89050	·89152	·89254	·89354	·89455	·89554	·89653	38°	10	20	30	40	50	60	70	80	90
52°	·89653	·89752	·89849	·89947	·90043	·90139	·90235	37°	10	19	29	39	49	58	68	78	87
53°	·90235	·90330	·90424	·90518	·90611	·90704	·90796	36°	9	19	28	37	47	56	65	74	84
54°	·90796	·90887	·90978	·91069	·91158	·91241	·91336	35°	9	18	27	36	45	54	63	72	81
55°	9·91336	9·91425	9·91512	9·91599	9·91686	9·91772	9·91857	34°	9	17	26	35	44	52	61	70	78
56°	·91857	·91942	·92027	·92111	·92194	·92277	·92359	33°	8	17	25	34	42	50	59	67	76
57°	·92359	·92441	·92522	·92603	·92683	·92763	·92842	32°	8	16	24	32	41	49	57	65	73
58°	·92842	·92921	·92999	·93077	·93154	·93230	·93307	31°	8	16	23	31	39	47	55	62	70
59°	·93307	·93382	·93457	·93532	·93606	·93680	·93753	30°	8	15	23	30	37	45	52	60	67
60°	9·93753	9·93826	9·93898	9·93970	9·94041	9·94112	9·94182	29°	7	14	22	29	36	43	50	57	64
61°	·94182	·94252	·94321	·94390	·94458	·94526	·94593	28°	7	14	21	27	34	41	48	55	62
62°	·94593	·94660	·94727	·94793	·94858	·94923	·94988	27°	7	13	20	26	33	40	46	53	59
63°	·94988	·95052	·95116	·95179	·95242	·95304	·95366	26°	6	13	19	25	32	38	44	50	57
64°	·95366	·95427	·95488	·95549	·95609	·95668	·95728	25°	6	12	18	24	30	36	42	48	54

65°	9·95728	9·95786	9·95844	9·95902	9·95960	9·96017	9·96073	24°	6	12	17	23	29	35	40	46	52
66°	·96073	·96129	·96185	·96240	·96294	·96349	·96403	23°	6	11	17	22	28	33	38	44	50
67°	·96403	·96456	·96509	·96562	·96614	·96665	·96717	22°	5	10	16	21	26	31	36	42	47
68°	·96717	·96767	·96818	·96868	·96917	·96966	·97015	21°	5	10	15	20	25	29	34	40	44
69°	·97015	·97063	·97111	·97159	·97206	·97252	·97299	20°	5	9	14	19	24	28	33	38	42
70°	9·97299	9·97344	9·97390	9·97435	9·97479	9·97523	9·97567	19°	4	9	13	18	22	27	31	36	40
71°	·97567	·97610	·97653	·97696	·97738	·97779	·97821	18°	4	9	13	17	21	26	30	34	38
72°	·97821	·97861	·97902	·97942	·97982	·98021	·98060	17°	4	8	12	16	20	24	28	32	36
73°	·98060	·98098	·98136	·98174	·98211	·98248	·98284	16°	4	8	11	15	19	22	26	30	34
74°	·98284	·98320	·98356	·98391	·98426	·98460	·98494	15°	4	7	11	14	18	21	25	28	32
75°	9·98494	9·98528	9·98561	9·98594	9·98627	9·98659	9·98690	14°	3	7	10	13	17	20	23	26	30
76°	·98690	·98722	·98753	·98783	·98813	·98843	·98872	13°	3	6	9	12	15	18	21	24	27
77°	·98872	·98901	·98930	·98958	·98986	·99013	·99040	12°	3	6	8	11	14	17	20	22	25
78°	·99040	·99067	·99093	·99119	·99145	·99170	·99195	11°	3	5	8	10	13	16	18	21	23
79°	·99195	·99219	·99243	·99267	·99290	·99313	·99335	10°	2	5	7	9	12	14	16	19	21
80°	9·99335	9·99357	9·99379	9·99400	9·99421	9·99442	9·99462	9°	2	4	6	8	11	13	15	17	19
81°	·99462	·99482	·99501	·99520	·99539	·99557	·99575	8°	2	4	6	8	10	11	13	15	17
82°	·99575	·99593	·99610	·99627	·99643	·99659	·99675	7°	2	3	5	7	8	10	12	13	15
83°	·99675	·99690	·99705	·99720	·99734	·99748	·99761	6°	1	3	4	6	7	9	10	12	13
84°	·99761	·99775	·99787	·99800	·99812	·99823	·99834	5°	1	3	4	5	6	8	9	10	11
85°	9·99834	9·99845	9·99856	9·99866	9·99876	9·99885	9·99894	4°	1	2	3	4	5	6	7	8	9
86°	·99894	·99903	·99911	·99919	·99926	·99934	·99940	3°	1	2	2	3	4	5	5	6	7
87°	·99940	·99947	·99953	·99959	·99964	·99969	·99974	2°	1	1	2	2	3	3	4	4	5
88°	·99974	·99978	·99982	·99985	·99988	·99991	·99993	1°	0	1	1	1	2	2	2	2	3
89°	·99993	·99995	·99997	·99998	·99999	10·00000	10·00000	0°									
90°	10·00000																
°	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

LOGARITHMIC COSINES

TABLE V

LOGARITHMIC TANGENTS

	0′	10′	20′	30′	40′	50′	60′		Mean Differences 1′	2′	3′	4′	5′	6′	7′	8′	9′
0°	$-\infty$	7·46373	7·76476	7·94086	8·06581	8·16273	8·24192	89°	Differences vary so rapidly here that tabulation is impossible. For small angles of x minutes log tan x or log cot $(90° - x')$ $= \log x + \bar{4}·46373$.								
1°	8·24192	8·30888	8·36689	8·41807	8·46385	8·50527	8·54308	88°									
2°	8·54308	8·57788	8·61009	8·64009	8·66816	8·69453	8·71940	87°									
3°	8·71940	8·74292	8·76525	8·78649	8·80674	8·82610	8·84464	86°									
4°	8·84464	8·86243	8·87953	8·89598	8·91165	8·92716	8·94195	85°									
5°	8·94195	8·95627	8·97013	8·98358	8·99662	9·00930	9·02162	84°									
6°	9·02162	9·03361	9·04528	9·05666	9·06775	9·07858	9·08914	83°									
7°	9·08914	9·09947	9·10956	9·11943	9·12909	9·13854	9·14780	82°	98	195	293	391	488	586	684	782	879
8°	9·14780	9·15688	9·16577	9·17450	9·18306	9·19146	9·19971	81°	87	173	260	346	433	519	606	692	779
9°	9·19971	9·20782	9·21578	9·22361	9·23130	9·23887	9·24632	80°	78	155	233	310	388	466	543	621	698
10°	9·24632	9·25365	9·26086	9·26797	9·27496	9·28186	9·28865	79°	71	141	212	282	354	420	494	564	635
11°	·28865	·29535	·30195	·30846	·31489	·32122	·32747	78°	65	129	194	259	323	388	453	518	582
12°	·32747	·33365	·33974	·34576	·35170	·35757	·36336	77°	60	120	179	239	299	359	419	478	538
13°	·36336	·36909	·37476	·38035	·38589	·39136	·39677	76°	56	111	167	222	278	334	389	445	500
14°	·39677	·40212	·40742	·41266	·41784	·42297	·42805	75°	52	104	156	208	261	313	365	417	469
15°	9·42805	9·43308	9·43806	9·44299	9·44787	9·45271	9·45750	74°	49	98	147	196	245	294	343	392	442
16°	·45750	·46224	·46694	·47160	·47622	·48080	·48534	73°	46	93	139	186	232	278	325	371	418
17°	·48534	·48984	·49430	·49872	·50311	·50746	·51178	72°	44	88	132	176	220	264	308	352	396
18°	·51178	·51606	·52031	·52452	·52870	·53285	·53697	71°	42	84	126	168	210	252	294	336	378
19°	·53697	·54106	·54512	·54915	·55315	·55712	·56107	70°	40	80	121	160	201	241	281	321	362

20°	9·56107	9·56498	9·56887	9·57274	9·57658	9·58039	9·58418	69°	39	77	116	154	193	231	270	308	347
21°	·58418	·58794	·59168	·59540	·59909	·60276	·60641	68°	37	74	111	148	185	222	259	296	333
22°	·60641	·61004	·61364	·61722	·62079	·62433	·62785	67°	36	72	107	143	179	214	250	286	322
23°	·62785	·63135	·63484	·63830	·64175	·64517	·64858	66°	35	69	104	138	173	208	242	277	311
24°	·64858	·65197	·65535	·65870	·66204	·66537	·66867	65°	34	67	101	134	168	201	235	268	302
25°	9·66867	9·67196	9·67524	9·67850	9·68174	9·68497	9·68818	64°	33	65	98	130	163	195	228	260	293
26°	·68818	·69138	·69457	·69774	·70089	·70404	·70717	63°	32	63	95	126	158	190	221	253	284
27°	·70717	·71028	·71339	·71648	·71955	·72262	·72567	62°	31	62	92	123	154	185	216	246	277
28°	·72567	·72872	·73175	·73476	·73777	·74077	·74375	61°	30	60	90	120	151	181	211	241	271
29°	·74375	·74673	·74969	·75264	·75558	·75852	·76144	60°	29	59	88	118	147	177	206	236	265
30°	9·76144	9·76435	9·76725	9·77015	9·77303	9·77591	9·77877	59°	29	58	87	116	144	173	202	231	260
31°	·77877	·78163	·78448	·78732	·79015	·79297	·79579	58°	28	57	85	113	142	170	198	227	255
32°	·79579	·79860	·80140	·80419	·80697	·80975	·81252	57°	28	56	84	112	139	167	195	223	251
33°	·81252	·81528	·81803	·82078	·82352	·82626	·82899	56°	28	55	83	110	137	165	192	220	247
34°	·82899	·83171	·83442	·83713	·83984	·84254	·84523	55°	27	54	81	108	136	162	190	217	244
35°	9·84523	9·84791	9·85059	9·85327	9·85594	9·85860	9·86126	54°	27	54	80	107	134	160	188	214	241
36°	·86126	·86392	·86656	·86921	·87185	·87448	·87711	53°	26	53	79	106	132	158	185	212	238
37°	·87711	·87974	·88236	·88498	·88759	·89020	·89281	52°	26	52	78	105	131	157	183	209	236
38°	·89281	·89541	·89801	·90061	·90320	·90578	·90837	51°	26	52	78	104	130	156	182	208	234
39°	·90837	·91095	·91353	·91610	·91868	·92125	·92381	50°	26	52	77	103	129	155	180	206	232
40°	9·92381	9·92638	9·92894	9·93150	9·93406	9·93661	9·93916	49°	26	51	77	102	128	154	179	205	230
41°	·93916	·94171	·94426	·94681	·94935	·95190	·95444	48°	25	51	76	102	127	153	178	204	229
42°	·95444	·95698	·95952	·96205	·96459	·96712	·96966	47°	25	51	76	101	127	152	177	203	228
43°	·96966	·97219	·97472	·97725	·97978	·98231	·98484	46°	25	51	76	101	127	152	177	202	228
44°	·98484	·98737	·98989	·99242	·99495	·99747	10·00000	45°	25	51	76	101	127	152	177	202	228
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

LOGARITHMIC COTANGENTS

LOGARITHMIC TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences								
45°	10·00000	10·00253	10·00505	10·00753	10·01011	10·01263	10·01516	44°	25	51	76	101	127	152	177	202	228
46°	·01516	·01769	·02022	·02275	·02528	·02781	·03034	43°	25	51	76	101	127	152	177	202	228
47°	·03034	·03288	·03541	·03795	·04043	·04302	·04556	42°	25	51	76	101	127	152	177	203	228
48°	·04556	·04810	·05065	·05319	·05574	·05829	·06084	41°	25	51	76	102	127	153	178	204	229
49°	·06084	·06339	·06594	·06850	·07106	·07362	·07619	40°	26	51	77	102	128	154	179	205	230
50°	10·07619	10·07875	10·08132	10·08390	10·08647	10·08905	10·09163	39°	26	52	77	103	129	155	180	206	232
51°	·09163	·09422	·09680	·09939	·10199	·10459	·10719	38°	26	52	78	104	130	156	182	208	234
52°	·10719	·10980	·11241	·11502	·11764	·12026	·12289	37°	26	52	78	105	131	157	183	209	236
53°	·12289	·12552	·12815	·13079	·13344	·13603	·13874	36°	26	53	79	106	132	158	185	212	238
54°	·13874	·14140	·14406	·14673	·14941	·15209	·15477	35°	27	54	80	107	134	160	188	214	241
55°	10·15477	10·15746	10·16016	10·16287	10·16558	10·16829	10·17101	34°	27	54	81	109	136	162	190	217	244
56°	·17101	·17374	·17648	·17922	·18197	·18472	·18748	33°	28	55	83	110	137	165	192	220	247
57°	·18748	·19025	·19303	·19581	·19860	·20140	·20421	32°	28	56	84	112	139	167	195	223	251
58°	·20421	·20703	·20985	·21268	·21552	·21837	·22123	31°	28	57	85	113	142	170	198	227	255
59°	·22123	·22409	·22697	·22985	·23275	·23565	·23856	30°	29	58	87	116	144	173	202	231	260
60°	10·23856	10·24148	10·24442	10·24736	10·25031	10·25327	10·25625	29°	29	59	88	118	147	177	206	236	265
61°	·25625	·25923	·26223	·26524	·26825	·27128	·27433	28°	30	60	90	120	151	181	211	241	271
62°	·27433	·27738	·28045	·28352	·28661	·28972	·29283	27°	31	62	92	123	154	185	216	246	277
63°	·29283	·29596	·29911	·30226	·30543	·30862	·31182	26°	32	63	95	126	158	190	221	253	284
64°	·31182	·31503	·31826	·32150	·32476	·32804	·33133	25°	33	65	98	130	163	195	228	260	293

	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′
65°	10·33133	10·33463	10·33796	10·34130	10·34465	10·34803	10·35142	24°	34	67	101	134	168	201	235	268	302
66°	·35142	·35483	·35825	·36170	·36516	·36865	·37215	23°	35	69	104	138	173	206	242	277	311
67°	·37215	·37567	·37921	·38278	·38636	·38996	·39359	22°	36	72	107	143	179	214	251	286	322
68°	·39359	·39724	·40091	·40460	·40832	·41206	·41582	21°	37	74	111	148	185	222	259	296	333
69°	·41582	·41961	·42342	·42726	·43113	·43502	·43893	20°	39	77	116	154	193	231	270	308	347
70°	10·43893	10·44288	10·44685	10·45085	10·45488	10·45894	10·46303	19°	40	80	121	160	201	241	281	321	362
71°	·46303	·46715	·47130	·47548	·47969	·48394	·48822	18°	42	84	126	168	210	252	294	336	378
72°	·48822	·49254	·49689	·50128	·50570	·51016	·51466	17°	44	88	132	176	220	264	308	352	396
73°	·51466	·51920	·52378	·52840	·53306	·53776	·54250	16°	46	93	139	186	232	278	325	371	418
74°	·54250	·54729	·55213	·55701	·56194	·56692	·57195	15°	49	98	147	196	245	294	343	392	442
75°	10·57195	10·57703	10·58216	10·58734	10·59258	10·59768	10·60323	14°	52	104	156	208	261	313	365	417	469
76°	·60323	·60864	·61411	·61965	·62524	·63091	·63664	13°	56	111	167	222	278	334	389	445	500
77°	·63664	·64243	·64830	·65424	·66026	·66635	·67253	12°	60	120	179	239	299	359	419	478	538
78°	·67253	·67878	·68511	·69154	·69805	·70465	·71135	11°	65	129	194	259	323	388	453	518	582
79°	·71135	·71814	·72504	·73203	·73914	·74635	·75368	10°	71	141	212	282	354	420	494	564	635
80°	10·75368	10·76113	10·76870	10·77639	10·78422	10·79218	10·80029	9°	78	155	233	310	388	466	543	621	698
81°	·80029	·80854	·81694	·82550	·83423	·84312	·85220	8°	87	173	260	346	433	519	606	692	779
82°	·85220	·86146	·87091	·88057	·89044	·90053	·91086	7°	98	195	293	391	488	586	684	782	879
83°	·91086	·92142	·93225	·94334	·95472	·96639	·97838	6°									
84°	·97838	·99070	11·00338	11·01642	11·02987	11·04373	11·05805	5°									
85°	11·05805	11·07284	11·08815	11·10402	11·12047	11·13757	11·15536	4°									
86°	11·15536	11·17390	11·19326	11·21351	11·23475	11·25708	11·28060	3°									
87°	11·28060	11·30547	11·33184	11·35991	11·38991	11·42212	11·45692	2°									
88°	11·45692	11·49473	11·53615	11·58193	11·63311	11·69112	11·75808	1°	Differences vary so rapidly here that								
89°	11·75808	11·83727	11·93419	12·05914	12·23524	12·53627	+∞	0°	tabulation is impossible.								
90°	+∞																
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

LOGARITHMIC COTANGENTS

SOME USEFUL CONSTANTS

One radian $= 57° \, 17' \, 45''$ nearly $= 206265''$;

$\log 206265 = 5{\cdot}3144255.$

$\pi = 3{\cdot}14159265\ldots$ $\quad \dfrac{1}{\pi} = 0{\cdot}31830989\ldots$

$\sqrt{2} = 1{\cdot}4142135\ldots$ $\quad \sqrt{3} = 1{\cdot}7320508\ldots$

$\sqrt{5} = 2{\cdot}2360679\ldots$ $\quad \sqrt{6} = 2{\cdot}4494897\ldots$

$\sqrt{7} = 2{\cdot}6457513\ldots$ $\quad \sqrt{8} = 2{\cdot}8284271\ldots$

$\sqrt{10} = 3{\cdot}1622776\ldots$

SOME USEFUL LOGARITHMS

$\log 2 = {\cdot}30103$ $\quad \log 3 = {\cdot}47712$

$\log 5 = {\cdot}69897$ $\quad \log 7 = {\cdot}84510$